16.6 Parameterizing Surfaces

Recall that $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ with $a \leq t \leq b$ gives a parameterization for a curve C. In section 16.2-16.4, we learned how to make measurements along curves for scalar and vector fields by using line integrals "$\int_C$". We computed these line integrals by first finding parameterizations (unless special theorems apply).

In a similar way, we will parameterize a surface S using

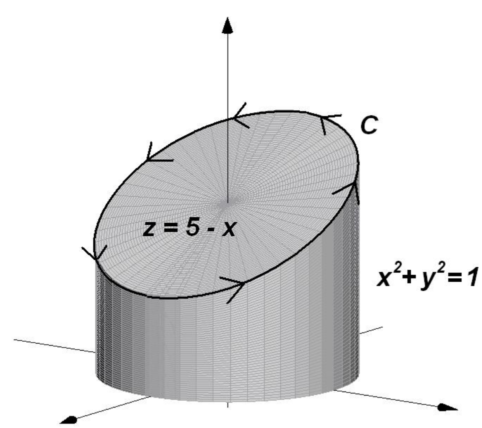
$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle,$$

where (u,v) are constrained to some region D in the uv-plane. In section 16.7-16.9, we learned how to make measurements across surfaces for scalar and vector fields by using surface integrals "$\iint_S$". We will compute these surface integrals by first finding parameterizations (and later we will learn theorems that apply in special cases).

For now, let's focus on parameterization.

Questions: Find a parameterization for each surface:

1. The part of the surface $z = 10$ that is above the square $-1 \leq x \leq 1$, $-2 \leq y \leq 2$.
2. The part of the surface $x - y + z = 4$ that is within the cylinder $x^2 + y^2 = 9$.
3. The part of the surface $z = x^2 + y^2$ that is above the region in the xy-plane given by $0 \leq x \leq 1$, $0 \leq y \leq x^2$.
4. The part of the paraboloid $y = 9 - x^2 - z^2$ that is on the positive y side of the xz-plane.
5. The part of the circular cylinder $x^2 + y^2 = 4$ that is between the planes $z = 1$ and $z = 5$.
6. The upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$.
7. The entire sphere $x^2 + y^2 + z^2 = 16$.
8. The surface of revolution given by rotating the region bounded by $y = x^3$ for $0 \leq x \leq 2$ about the x-axis.
9. Find the parameterization for all three sides of the solid object within $x^2 + y^2 = 1$, above $z = 0$ and below $z = 5 - x$ shown here (ignore the curve):



Solutions:

1. Notes: The parameterization is already given!
   $\mathbf{r}(u,v) = \langle u, v, 10\rangle$, (I am just letting $x = u$ and $y = v$).
   You could also just leave them as x and y and give the parameterization as:
   $\mathbf{r}(x,y) = \langle x, y, 10\rangle$ with $-1 \le x \le 1$, $-2 \le y \le 2$.

2. Notes: The surface can easily be solve for z in terms of x and y.
   $\mathbf{r}(u,v) = \langle u, v, 4 - u + v\rangle$, (Letting $x = u$ and $y = v$, again). Also can be written as:
   $\mathbf{r}(x,y) = \langle x, y, 4 - x + y\rangle$ for points (x,y) inside the circular region $x^2 + y^2 \le 4$ (which we will do with polar when we get to the integral).

3. $\mathbf{r}(x,y) = \langle x, y, x^2 + y^2\rangle$ for points (x,y) inside the region given by $0 \le x \le 1$, $0 \le y \le x^2$ (again, we will account for this in the integral later).

4. Notes: This time it is easiest to give y in terms of x and z.
   $\mathbf{r}(x,z) = \langle x, 9 - x^2 - z^2, z\rangle$ for points (x,z) within the region when $y \ge 0$ on the surface. That would be when $9 - x^2 - z^2 \ge 0$ which would be the circular region $x^2 + z^2 \le 9$.

5. Notes: This is different from the previous cases, because one variable is 'missing' from the surface we wish to describe. That means z can be anything and we should make it one of our parameters. Then we need to find a parameterization for the other two variables. Look to use Sine and Cosine!
   $\mathbf{r}(u,v) = \langle 2\cos(u), 2\sin(u), v\rangle$, (This time, I am letting $x = 2\cos(u)$, $y = 2\sin(u)$ and $z = v$).
   We need $1 \le v \le 5$ from the given condition.
   And we need $0 \le u \le 2\pi$ to go all the way around the cylinder.

6. Notes: This could be done in a couple ways. Here are two different parameterizations:

   (a) We could just get z in terms of x and y. That would give $z = \sqrt{9 - x^2 - y^2}$ for the upper hemisphere. Giving the parameterization
   $\mathbf{r}(x,y) = \langle x, y, \sqrt{9 - x^2 - y^2}\rangle$, where (x,y) come from the region that corresponds to $z \ge 0$ in the surface equation, so $9 - x^2 - y^2 \ge 0$, which is the circular region $x^2 + y^2 \le 9$.

   (b) We could use spherical coordinators. Notice that the radius of the sphere, $\rho = 3$, is fixed.
   $\mathbf{r}(\phi,\theta) = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi\rangle$, where (ϕ,θ) satisfy $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$.

7. Notes: I would use spherical coordinates here (or break the problem into two parts; upper and lower hemisphere). Again the radius of the sphere, $\rho = 4$, is fixed.
   $\mathbf{r}(\phi,\theta) = \langle 4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi\rangle$, where (ϕ,θ) would satisfy $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.

8. Notes: For a surface of revolution about the x-axis, there is a circle of radius $f(x)$ about each value of x. So we can parameterize each of those circles to get
   $\mathbf{r}(u,v) = \langle u, f(u)\cos(v), f(u)\sin(v)\rangle$, so I am just replacing $x = u$ and then paramterizing the circle. The range of values would be $0 \le u \le 2$, and $0 \le v \le 2\pi$.

9. Here is a parameterization for each side:

   (a) Bottom: $\mathbf{r}(x,y) = \langle x, y, 0\rangle$, where (x,y) are in the region $x^2 + y^2 \le 1$.

   (b) Top: $\mathbf{r}(x,y) = \langle x, y, 5 - x\rangle$, where (x,y) are in the region $x^2 + y^2 \le 1$.

   (c) Sides: $\mathbf{r}(u,v) = \langle \cos(u), \sin(u), v\rangle$, where (u,v) satisfy $0 \le u \le 2\pi$ and $0 \le v \le 5 - \cos(u)$. (I got the last bound because z is always between 0 and $5 - x$ and in this parameterization $z = v$ and $x = \cos(u)$).

Surface Area

After parameterizing, our next step will be to give an expression for surface area.

Way back in 15.6, we already learned that the surface area for a surface parameterized by $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$ over a region D is given by $\iint_D 1\, dS$, where

$$dS = |r_x \times r_y| dA = \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA.$$

That was only for those particular parameterizations.

But the same general analysis applies. For a parameterization, $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$. We have

$\mathbf{r}_u = \langle x_u, y_u, z_u\rangle$ = a tangent vector to the surface in the u-direction.
$\mathbf{r}_v = \langle x_v, y_v, z_v\rangle$ = a tangent vector to the surface in the u-direction.

We then get several facts:

1. $\mathbf{r}_u$ and $\mathbf{r}_v$ together determine the tangent plane at a given point (because they are both 'on' this plane). So
   $\mathbf{r}_u \times \mathbf{r}_v$ would be a normal vector for the surface at a given point (and a normal for the tangent plane at that point).

2. If a small change in u and a small change in v are made, Δu and Δv, respectively, then we can estimate the resulting change in surface area by

$$\Delta S = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v.$$

   As Δu and Δv go to zero, this gets more precise and we write the surface area differential for this relationship as

$$dS = |\mathbf{r}_u \times \mathbf{r}_v| du dv.$$

3. From 15.6, the surface area of the surface is given by

$$\text{Surface area } = \iint_D dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

4. Some shortcuts:

   (a) For a parameterization of the form $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$, we get

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x, -f_y, 1\rangle$$

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{(f_x)^2 + (f_y)^2 + 1}$$

   (b) For a parameterization of the form $\mathbf{r}(\phi, \theta) = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi\rangle$, we get

$$\mathbf{r}_x \times \mathbf{r}_y = \langle a^2 \sin^2\phi \cos\theta, a^2 \sin^2\phi \sin\theta, a^2 \cos^2\phi\rangle$$

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin\phi$$