Integrals Review

Scalar Fields: Assume we have a real valued function $f(x,y)$ or $f(x,y,z)$ in $\Re^2$ or $\Re^3$, respectively. If we wish to measure (*add up*) the values over a line, curve, region, or solid, then we are using some type of integral. For this exercise, let's focus on two applications:

1. If $\rho(x,y)$ or $\rho(x,y,z)$ gives the density (in units of kg/m, kg/m^2, or kg/m^3 depending on the application), then the integral of ρ over some set gives the total mass of that set.
   - $\iint\limits_D \rho(x,y)\,dA =$ total mass of the region D in $\Re^2$
   - $\iiint\limits_E \rho(x,y,z)\,dV =$ total mass of the solid region E in $\Re^3$
   - $\int_C \rho(x,y)\,ds =$ total mass of the curve C in $\Re^2$
   - $\int_C \rho(x,y,z)\,ds =$ total mass of the curve C in $\Re^3$
   - $\iint\limits_S \rho(x,y,z)\,dS =$ total mass of the surface S in $\Re^3$

2. If $f(x,y)$ or $f(x,y,z)$ gives some value (think temperature), then the integral of f divided by the length, area, or volume (as appropriate) gives the average value of f over the set.
   - $\dfrac{1}{\text{Area of } D}\iint\limits_D f(x,y)\,dA =$ the average value of f over D
   - $\dfrac{1}{\text{Volume of } E}\iiint\limits_E f(x,y,z)\,dV =$ the average value of f over E
   - $\dfrac{1}{\text{length of } C}\int_C f(x,y)\,ds =$ the average value of f over C
   - $\dfrac{1}{\text{length of } C}\int_C f(x,y,z)\,ds =$ the average value of f over C
   - $\dfrac{1}{\text{surface area of } S}\iint\limits_S f(x,y,z)\,dS =$ the average value of f over S

Recall that:

1. $ds = |\mathbf{r}'(t)|dt$,
   in $\Re^2$, this becomes $ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$,
   in $\Re^3$, this becomes $ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt$.
   And $\int_C 1\,ds$ gives the arc length of the curve, C.

2. $dS = |\mathbf{r}_u \times \mathbf{r}_v|dudv$,
   And $\iint\limits_S 1\,dS$ gives the surface area of the surface, S.

Vector Fields: Assume we have a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ or $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ in $\Re^2$ or $\Re^3$, respectively. If we wish to measure (*add up*) the projection of the field vectors along or across a curve or surface, then we use integrals.

1. First if we want to measure the flow along a curve in a given direction, then recall that $\mathbf{F}\cdot\mathbf{T}$ gives the projection of $\mathbf{F}$ in the direction of the unit tangent $\mathbf{T}$ along the curve. We learned we could simplify this as follows:

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|dt = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt.$$

Then we started using the short-hand

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt.$$

This measure the flow or circulation of the vector field along the curve in the given direction. If the vectors $\mathbf{F}$ have units of force (Newtons), then this integral measures the work done by the field on an object moving along this path (in Joules).

2. Next if we want to measure the flow across a surface in a given normal direction, then we used the projection $\mathbf{F}\cdot\mathbf{n}$ where $\mathbf{n}$ is a unit normal to the surface. We learned we could simplify this as:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}|\mathbf{r}_u\times\mathbf{r}_v|dA = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.$$

Then we started using the short-hand

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.$$

This measure the flow of the vector field across the surface S in the given direction. If the vectors $\mathbf{F}$ are given in units of $kg/m^3\cdot m/s$ (density times velocity of the fluid), then this surface integral gives the flow in units of kg/s across the surface S.

Concerning orientation:

1. For a curve parameterized by $\mathbf{r}(t)$, $a\le t\le b$. You can figure out the orientation by plugging in $t=a$ (that would be the start) and $t=b$ (that would be the end). If you desire the opposite orientation, then just flip the sign of your output.

2. For a surface parameterized by $\mathbf{r}(u,v)$, you can figure out the orientation by looking at the components of $\mathbf{r}_u\times\mathbf{r}_v$ (which is the normal vector that the parameterization gives). For example, if the third component of $\mathbf{r}_u\times\mathbf{r}_v$ is positive, then the orientation is upward. If you desire the opposite orientation, then just flip the sign of your output.