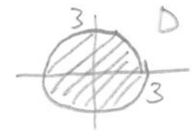
1. (10 pts) Find the surface area of the part of the paraboloid $z = 9 - x^2 - y^2$ that is above the xy-plane. (Set up AND evaluate).

- $\dfrac{\partial z}{\partial x} = -2x, \quad \dfrac{\partial z}{\partial y} = -2y$
- $SA = \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\, dA$, where D is the region inside the intersection of the paraboloid and the xy-plane $\Rightarrow 0 = 9 - x^2 - y^2 \Rightarrow x^2 + y^2 = 9$



- USE POLAR! $\quad 0 \le \theta \le 2\pi$, $\quad 0 \le r \le 3$

- $SA = \int_0^{2\pi} \int_0^3 \sqrt{4r^2+1}\; r\, dr\, d\theta$

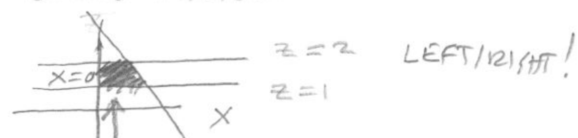
$u = 4r^2 + 1$
$du = 8r\,dr$
$dr = \frac{1}{8r} du$

$= 2\pi \int_1^{37} \sqrt{u}\, \frac{1}{8}\, du$

$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^{37} = \boxed{\frac{\pi}{6}\left[(37)^{3/2} - 1\right]}$

2. (7 pts) Consider the solid region E that is bounded by the planes $z = 1$, $z = 2$, $y = 0$, $x = 0$, and $3x + y + z = 9$. Set up a triple integral for the volume of the solid E (any one of the six possible orderings). DO NOT EVALUATE.

ONLY dx AND dy ARE REASONABLE INSIDE BOUND CHOICES
(dz WOULD BE A POOR CHOICE)

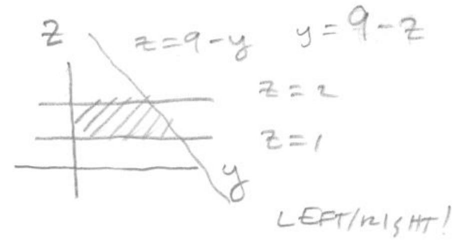


- $\boxed{dy \text{ INSIDE}}$ $\quad 0 \le y \le 9 - 3x - z$

INTERSECTION CURVE $\quad 0 = 9 - 3x - z$
$\Rightarrow z = 9 - 3x$

$z = 9 - 3x$
$x = \frac{1}{3}(9 - z) = 3 - \frac{1}{3} z$
ONLY REGION BOUNDED BY $z=1$ and $z=2$.
Here

$$\boxed{\int_1^2 \int_0^{3-\frac{1}{3}z} \int_0^{9-3x-z} 1\, dy\, dx\, dz}$$



- $\boxed{dx \text{ INSIDE}}$ $\quad 0 \le x \le 3 - \frac{1}{3}y - \frac{1}{3}z$

INTERSECTION CURVE
$0 = 3 - \frac{1}{3}y - \frac{1}{3}z$
$z = 9 - y$

$$\boxed{\int_1^2 \int_0^{9-z} \int_0^{3-\frac{1}{3}y-\frac{1}{3}z} 1\, dx\, dy\, dz}$$

3. (9 pts) Let E be the solid region that is bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the plane $z = 1$. The density of the solid is proportional to the distance from the z-axis. Using cylindrical coordinates, set up the integral formula for the z-coordinate of the center of mass. (Set up all bounds and integrands in cylindrical coordinates, but DO NOT EVALUATE).

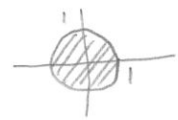
- $\rho(x,y,z) = K\underbrace{\sqrt{x^2+y^2}}$ DIST FROM (x,y,z) TO $(0,0,z)$

IN CYLINDRICAL COORDINATES $\boxed{\rho(x,y,z) = kr}$

- $\boxed{dz \text{ INSIDE}}$ $\sqrt{x^2+y^2} \leq z \leq 1 \Rightarrow \boxed{r \leq z \leq 1}$

REGION OF INTEGRATION IS INSIDE INTERSECTION CURVE

$1 = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = 1$



$$\boxed{\begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \end{array}}$$

- THUS,

$$\bar{z} = \frac{\iiint_E z\,\rho(x,y,z)\,dV}{\iiint_E \rho(x,y,z)\,dV}$$

$$\boxed{\bar{z} = \frac{\int_0^{2\pi}\int_0^1\int_r^1 z\,kr\,r\,dz\,dr\,d\theta}{\int_0^{2\pi}\int_0^1\int_r^1 kr\,r\,dz\,dr\,d\theta}}$$

4. (12 pts) If we label the center of the planet Nepluto as the origin, the temperature at a point (x, y, z) is given by the formula $T(x, y, z) = \dfrac{1000}{\sqrt{x^2 + y^2 + z^2}}$ in degrees Celcius, where x, y, and z, are in miles. The *mantle* of Nepluto is the inner layer of the planet that is between the radii of 3 and 5 miles from the center.

Using a triple integral in spherical coordinates, find the average temperature in mantle layer of the planet Nepluto. (In case you don't remember: The volume of a sphere is $\frac{4}{3}\pi r^3$.)

- $\text{AVE. TEMP.} = \dfrac{1}{\text{VOLUME}} \iiint_E T(x,y,z)\,dV$

- $\text{VOLUME} = \frac{4}{3}\pi(5)^3 - \frac{4}{3}\pi(3)^3 = \frac{4}{3}\pi(125-27) = \frac{4}{3}\pi \cdot 98 \text{ miles}^3$

- $\iiint_E T(x,y,z)\,dV = \int_0^{\pi}\int_0^{2\pi}\int_3^5 \dfrac{1000}{\rho}\,\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$

$= 1000 \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_3^5 \rho\, d\rho$

$= 1000 \left(\underbrace{-\cos\phi \Big|_0^{\pi}}_{\underbrace{--1--1}_{2}} \right) 2\pi \left(\frac{1}{2}\rho^2 \Big|_3^5 \right)$

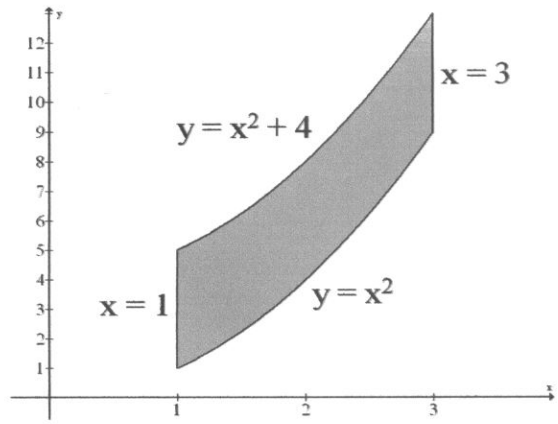
$= 4000\pi \cdot \frac{1}{2}(5^2 - 3^2)$

$= 2000\pi \cdot 16 = 32000\pi$

$$\boxed{\text{AVE. TEMP} = \frac{32000\cancel{\pi}}{\frac{4}{3}\cancel{\pi}\cdot 98} = \frac{24000}{98} = \frac{12000}{49} \approx 244.9\ ^\circ C}$$

5. (12 pts) Let R be the region bounded by $x = 1$, $x = 3$, $y = x^2$ and $y = x^2 + 4$ in $\mathbf{R}^2$ as shown below. Compute the value of $\displaystyle\iint_R \frac{x}{1 + y - x^2}\, dA$ by using the transformation: $x = \sqrt{u}$, $y = v + u$.

(You must use the transformation, including finding the image and the Jacobian, for full credit).



- $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{u}} & 0 \\ 1 & 1 \end{vmatrix} = \dfrac{1}{2\sqrt{u}}$

- IMAGE

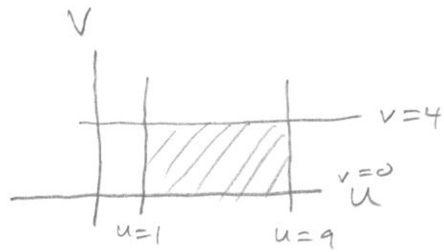
A $x = 1 \Rightarrow 1 = \sqrt{u} \Rightarrow u = 1$

B $x = 3 \Rightarrow 3 = \sqrt{u} \Rightarrow u = 9$

C $y = x^2 \Rightarrow v + u = u \Rightarrow v = 0$

D $y = x^2 + 4 \Rightarrow v + u = u + 4 \Rightarrow v = 4$

NOTE: $y - x^2 = v$ (which is how I picked v)



$$\iint_R \frac{x}{1+y-x^2}\, dA = \int_0^4 \int_1^9 \frac{\sqrt{u}}{1+v} \frac{1}{2\sqrt{u}}\, du\, dv$$

$$= \frac{1}{2} \int_0^4 \frac{1}{1+v}\, dv \underbrace{\int_1^9 du}_{= 9-1 = 8}$$

$$= 4 \ln|1+v| \Big|_0^4 = \boxed{4\ln(5) \approx 6.4378}$$