1. (13 pts) Felix is walking on the surface $z = f(x,y) = \frac{1}{4}x^2 + \frac{1}{3}y^3 - \frac{1}{2}xy^2$, where x, y and z are in miles. Label the positive y-direction as NORTH and the positive x-direction as EAST.

(a) Felix's x and y coordinates are given by $x = x(t)$ and $y = y(t)$ where t is in hours since he started walking. We are told that $x(1) = 2$, $y(1) = 1$, $x'(1) = 3$ and $y'(1) = -4$.
At $t = 1$ hour, find the rate of change of Felix's height with respect to time.
(Give units for your answer)

CHAIN RULE!

$$\frac{\partial z}{\partial x} = \frac{1}{2}x - \frac{1}{2}y^2 \qquad \Rightarrow \qquad \frac{dz}{dt} = \left(\frac{1}{2}x - \frac{1}{2}y^2\right)\frac{dx}{dt} + (y^2 - xy)\frac{dy}{dt}$$

$$\frac{\partial z}{\partial y} = y^2 - xy \qquad x = 2, y = 1, x' = 3, y' = -4$$

$$\Rightarrow \frac{dz}{dt} = \left(1 - \frac{1}{2}\right)(3) + (1-2)(-4)$$

$$= \frac{3}{2} + 4 = \boxed{\frac{11}{2} \quad \frac{\text{miles}}{\text{hour}}}$$

(b) Later Felix stops and takes a break at the point $(x,y) = (8,2)$. Give the **unit** direction vector that points in the (x,y) direction Felix needs to initially walk in order to go steepest **downhill**.

STEEPEST UPHILL $\Leftrightarrow$ GRADIENT!

$$\nabla f(8,2) = \left\langle \frac{1}{2}(8) - \frac{1}{2}(2)^2, (2)^2 - (8)(2) \right\rangle = \langle 4-2, 4-16 \rangle = \langle 2, -12 \rangle$$

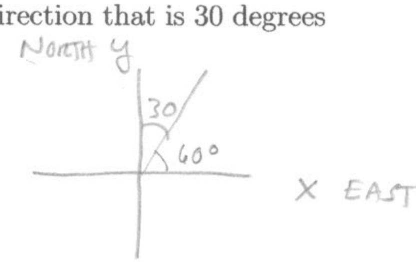
STEEPEST DOWNHILL $\Rightarrow$ $\langle -2, 12 \rangle$ DIRECTION

UNIT VECTOR $\boxed{\frac{1}{\sqrt{4+144}}\langle -2, 12 \rangle = \frac{1}{\sqrt{148}}\langle -2, 12 \rangle = \frac{1}{\sqrt{37}}\langle -1, 6 \rangle}$

(c) Felix is still standing at $(x,y) = (8,2)$. He decides to walk in the direction that is 30 degrees east of north. Find the slope in this direction.

$$\vec{u} = \langle \cos(60°), \sin(60°) \rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

DIRECTION:



$$D_{\vec{u}} f(8,2) = \nabla f(8,2) \cdot \vec{u}$$

$$= \langle 2, -12 \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$= \boxed{1 - 6\sqrt{3}}$$

2. (12 pts) Consider the vector field $\mathbf{F}(x, y) = \langle xy, -x^2y \rangle$ on $\mathbb{R}^2$.
Note: For any computations that require a z-component, assume the z-component is zero.

(a) Compute curl $\mathbf{F}$.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -x^2y & 0 \end{vmatrix} = (0-0)\vec{i} - (0-0)\vec{j} + (-2xy - x)\vec{k}$$

$$= \boxed{(-2xy - x)\vec{k} = \langle 0, 0, -2xy - x \rangle}$$

(b) Compute div $\mathbf{F}$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

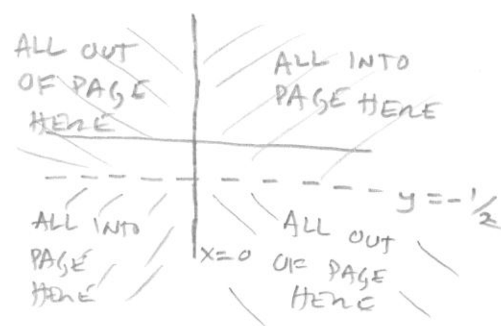
$$= \boxed{y - x^2 + 0 = y - x^2}$$

(c) Give an example (any example) of a point (x_0, y_0) in the vector field at which the curl $\mathbf{F}$ points in the negative z direction (into the page). And tell me if the vector field has a tendency to rotate clockwise or counterclockwise about the point you have given.

$\text{curl } \vec{F} = \vec{0}$ when $-2xy - x = 0 \iff -x(2y+1) = 0 \iff x = 0$ or $y = -\frac{1}{2}$

$\boxed{\text{NEEDS} \quad x > 0, y > -\tfrac{1}{2} \quad \text{OR} \quad x < 0, y < -\tfrac{1}{2}}$

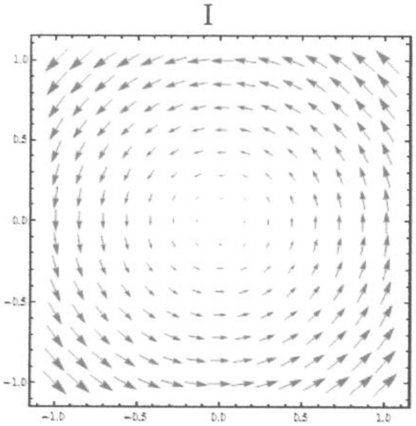
ANY SUCH EXAMPLE



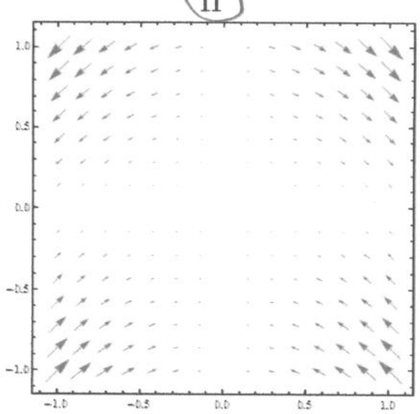
Circle one: Clockwise or Counterclockwise.

(d) Circle the picture that corresponds to this vector field.

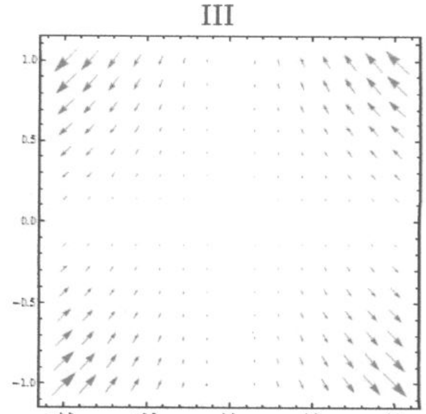
I



II



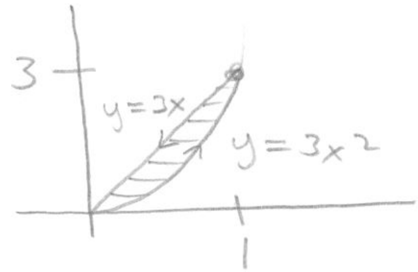
III



3. (a) (7 pts) Let C be the closed loop given by first following $y = 3x^2$ from $(0,0)$ to $(1,3)$, then following the straight line back from $(1,3)$ to $(0,0)$.

Using any appropriate method, evaluate $\oint_C (2xy^2 + \sin(x))\,dx + (y - y^3)\,dy$.

GREEN'S THM!



$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_D 0 - 4xy\,dA$$

$$= \int_0^1 \int_{3x^2}^{3x} -4xy\,dy\,dx$$

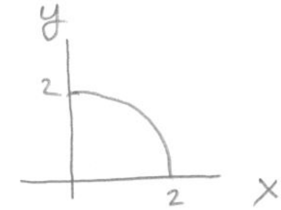
$$= \int_0^1 -2xy^2 \Big|_{3x^2}^{3x} dx$$

$$= \int_0^1 -18x^3 - -18x^5\,dx$$

$$= \frac{-18}{4}x^4 + \frac{18}{6}x^6 \Big|_0^1 = -\frac{9}{2} + 3 = \boxed{-\frac{3}{2}}$$

(b) (6 pts) The base of a fence is the circle of radius 2 in the first quadrant. The height of the fence at position (x, y) is given by the function $h(x, y) = 3 + x$. Give the area of one side of the fence.

PARAMETERIZE!



$x = 2\cos(t),\ y = 2\sin(t) \qquad 0 \le t \le \pi/2$

$ds = \sqrt{(-2\sin(t))^2 + (2\cos(t))^2}\,dt = 2\,dt$

$$\text{AREA} = \int_C 3 + x\,ds$$

$$= \int_0^{\pi/2} (3 + 2\cos(t))\,2\,dt = 6t + 4\sin(t)\Big|_0^{\pi/2}$$

$$= \boxed{3\pi + 4 \quad \text{square units}}$$

4. (12 pts) Consider the vector field $\mathbf{F} = \langle -y^2 \sin(xy^2), -2xy\sin(xy^2) + 2e^{2y}, \sqrt[3]{z} \rangle$.
This vector field is conservative!

(a) Find a function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

① $f(x,y,z) = \int -y^2 \sin(xy^2)\,dx = \cos(xy^2) + g(y,z)$

② $f_y(x,y,z) = -2xy\sin(xy^2) + g_y(y,z) \stackrel{?}{=} -2xy\sin(xy^2) + 2e^{2y}$

$\Rightarrow f(x,y,z) = \cos(xy^2) + \int 2e^{2y}\,dy = \cos(xy^2) + e^{2y} + h(z)$

③ $f_z(x,y,z) = 0 + 0 + h'(z) \stackrel{?}{=} z^{1/3}$

$\Rightarrow f(x,y,z) = \cos(xy^2) + e^{2y} + \int z^{1/3}\,dz$

$\boxed{f(x,y,z) = \cos(xy^2) + e^{2y} + \frac{3}{4} z^{4/3} + C}$

any constant here would work

(b) Let C_1 be the line from $(0, 0, 0)$ to $(1, 2, 8)$ and
let C_2 be the curve parameterized by $x = 1 + t - t^2$, $y = 3 - 3t^4$, $z = 8 + \sin(\pi t)$ for $0 \le t \le 1$.
Let C be the curve given by C_1 followed by C_2.
Using any appropriate method, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Please use your fastest option!).

START : $(0,0,0)$

END : $(1+1-1, 3-3, 8+0) = (1,0,8)$

$\int_C \vec{F} \cdot d\vec{r} = f(1,0,8) - f(0,0,0)$

$= \left[\cos(0) + e^0 + \frac{3}{4} 8^{4/3}\right] - \left[\cos(0) + e^0 + \frac{3}{4} 0^{4/3}\right]$

$= 2 + \frac{3}{4} 2^4 - 2$

$= \boxed{12}$