16.7: Surface Integrals

In this section we define the surface integral of scalar field and of a vector field as:

$$\iint_S f(x, y, z)\, dS \quad \text{and} \quad \iint_S \mathbf{F} \cdot d\mathbf{S}.$$

For both definitions, we start with a surface, S, that is parameterized by

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k},$$

with (u, v) over some domain D. In both cases, we evaluate the integral via:

1. Parameterizing
2. Computing $\mathbf{r}_u \times \mathbf{r}_v$. (which gives a normal vector for the surface)
3. Replacing $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$.
4. Finding the region D and computing the resulting double integral.

The two integrals are computed via:

$$\iint_S f(x, y, z)\, dS = \iint_D f(x(u, v), y(u, v), z(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$$

and the surface integral of a vector field

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA \\ &= \iint_D \langle P(x(u, v), y(u, v), z(u, v)), Q(x(u, v), y(u, v), z(u, v)), R(x(u, v), y(u, v), z(u, v))\rangle \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA\end{aligned}$$

Quick Basic Examples:

1. Consider the surface, S, given by $r(u, v) = 3u\mathbf{i} + 5v\mathbf{j} + (v^2 + u)\mathbf{k}$ with $0 \le u \le 2$ and $1 \le v \le 3$.
2. Then $\mathbf{r}_u \times \mathbf{r}_v = \langle -5, -6v, 15\rangle$ (you should know how to compute this, check!).
3. If you were given a scalar function $f(x, y, z) = xz + y^2$, the integral would be

$$\iint_S xz + y^2\, dS = \int_1^3 \int_0^2 ((3u)(v^2 + u) + (5v)^2) \sqrt{(-5)^2 + (-6v)^2 + (15)^2}\, dudv$$

4. If you were given a vector function $\mathbf{F}(x, y, z) = \langle z, xy, y^2\rangle$, the integral would be

$$\begin{aligned}\iint_S \langle z, xy, y^2\rangle \cdot d\mathbf{S} &= \int_1^3 \int_0^2 \langle v^2 + u, (3u)(5v), (5v)^2\rangle \cdot \langle -5, -6v, 15\rangle\, dudv \\ &= \int_1^3 \int_0^2 -5(v^2 + u) - 6v(3u)(5v) + 15(5v)^2\, dudv\end{aligned}$$

Special Cases/Time Savers: The second case below covers a large number of situations, so you don't always have to go through the full process of parameterization.

(a) **SPHERE**: If we parameterize a sphere using spherical coordinates we have $\mathbf{r}(\phi, \theta) = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi \rangle$ we found that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle a^2 \sin^2\phi \cos\theta, a^2 \sin^2\phi \sin\theta, a^2 \cos\phi \sin\phi \rangle \quad \text{and} \quad |\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin\phi.$$

(b) **GRAPH**: If we are given a surface in the form $z = g(x, y)$, then we can use the quick parameterization $\mathbf{r}(x, y) = \langle x, y, g(x, y) \rangle$ and we found that

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \rangle \quad \text{and} \quad |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}.$$

Orientation: For the integral $\iint\limits_S f(x, y, z)\, dS$, orientation doesn't matter. We don't even have to think about it. But for the integral $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$ orientation does matter. By orientation we mean what direction are the normal vectors pointing. In order for this to make sense, we must be working with two sided surfaces (no Mobius strips in this class).

(a) If $\mathbf{r}_u \times \mathbf{r}_v = \langle BLAH, STUFF, THING \rangle$, then we say

   i. If $THING$ is positive, the orientation is upward (in direction of positive z-axis). If $THING$ is negative, the orientation is downward.

   ii. If $STUFF$ is positive, the orientation is in the direction of the positive y-axis. If $STUFF$ is negative, the orientation is in the direction of the negative y-axis.

   iii. If $BLAH$ is positive, the orientation is in the direction of the positive x-axis. If $BLAH$ is negative, the orientation is in the direction of the negative x-axis.

(b) Sometime we talk of *outward* and *inward* orientation for spheres, cylinders or **closed** surfaces. The convention is, that when we use the term **positive** orientation for a closed service, we mean outward orientation. The parameterization for a sphere given early is an outward orientation.

(c) If you have the wrong orientation, you can get a normal vector that points directly in the opposite direction from $-\mathbf{r}_u \times \mathbf{r}_v$, which means the oppositive orientation has the opposite sign. That is, if S_1 is parameterized with upward (or outward) orientation and S_2 is the same surface with downward (or inward) orientation, then

$$\iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S} = -\iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$