16.8: Stokes Theorem and 16.9: Divergence Theorem

In these two sections we gave two generalizations of Green's theorem. These theorems relate measurements on a region to measurements on the regions boundary. In that sense, they are theorems similar in style to the fundamental theorem of calculus. They also provide information about the purpose and meaning of the $\operatorname{curl}\mathbf{F}$ and $\operatorname{div}\mathbf{F}$.

Stoke's Theorem:
Let S be an oriented surface with a simple, closed boundary C. We use the **positive** orientation for C, meaning as you walk around C the surface is on your left and your head is in the direction of the normal for S. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S},$$

which is the same as saying (from the original definitions of these integrals):

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS.$$

1. Stoke's theorem is useful precisely in a situation when you have better information about the boundary curve than you do the surface (or vice versa). It is not typically my first thought to try and use Stoke's theorem unless I am in one of the following two particular situations:

   (a) I have a closed curve C in $\mathbf{R}^3$. I want to compute $\int_C \mathbf{F}\cdot d\mathbf{r}$. For whatever reason it is difficult or time consuming to parameterize C. And there is a simple surface S that has C as it's boundary. Or the $\operatorname{curl}\mathbf{F}$ is nicer or easier to work with that $\mathbf{F}$.

   (b) I have a surface S. I want to compute $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$. For whatever reason it is difficult or time consuming to parameterize S. And there is a simple boundary C for S that is easy to parameterize. Or the $\mathbf{F}$ is nicer or easier to work with that $\operatorname{curl}\mathbf{F}$

2. Note the orientation comments in the theorem. For example: If S is has upward orientation, then C will need to have counterclockwise orientation when viewed from above. So you do need to check orientation when you do this (or else you might get the sign off).

3. Notice that it is surface independent in the following sense: Given a closed curve C in $\mathbf{R}^3$, you can compute the surface integral of $\operatorname{curl}\mathbf{F}$ over any surface that has C as the boundary.

4. If you recall in 16.5, we noted that Green's Theorem can be written as:

   $$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dA$$

   So in the special case, when the vector field is on the plane, meaning $\mathbf{F}(x,y,z) = \langle P(x,y), Q(x,y), 0\rangle$, Stoke's theorem becomes Green's Theorem (so you can now forget Green's theorem and only remember Stoke's theorem if you want, ha).

5. We had a discussion in class (and it is in the the book on pate 1096), about how this theorem helps us to understand what $\operatorname{curl}\mathbf{F}$ represents. How it gives the axis for the greatest rotating effect and how it measure the circulation about that axis at a point.

Divergence Theorem:
Let E be a simple solid region and let S be the boundary of E, given with positive (outward) orientation. Then

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iiint\limits_E \operatorname{div} \mathbf{F} \, dV.$$

1. Typically, we use the Divergence theorem in a situation where we are given a closed surface and we want to compute $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$. For whatever reason it is difficult or time consuming to parameterize S, but it is easier to set up a triple integral over the solid E.

2. Note again orientation comments in the theorem. If S is oriented inward, instead of outward, then you have $\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = -\iiint\limits_E \operatorname{div} \mathbf{F} \, dV$.

3. If you recall in 16.5, we noted that Green's Theorem could be used to show that the normal component field across C is given by

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint\limits_D \operatorname{div} \mathbf{F} dA.$$

   So in the special case, when the vector field is on the plane, meaning $\mathbf{F}(x, y, z) = \langle P(x, y), Q(x, y), 0 \rangle$, The divergence theorem becomes the flux across a curve version of Green's Theorem.

4. We had a discussion in class (and it is in the the book on pate 1103), about how this theorem helps us to understand what $\operatorname{div} \mathbf{F}$ represents. How it gives the net rate of outward flux per unit volume, so if $\operatorname{div} \mathbf{F}(P)$ is positive, then the point P is a **source** and if $\operatorname{div} \mathbf{F}(P)$ is negative, then the point P is a **sink**.