16.2: Line Integrals

In this section as well as in 16.3 and 16.4, we start with a curve C in $\mathbb{R}^2$ or $\mathbb{R}^3$.
We find, or are given, a parameterization of the curve in the form $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ in $\mathbb{R}^2$ or $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ in $\mathbb{R}^3$ for some interval $a \le t \le b$. Then we compute an integral over the curve.

IN BRIEF There are various types of integrals over the curve that we will be interested in. But really this section is about the two integrals:
The line integral with respect to arc length of a scalar function

$$\int_C f(x,y)\,ds \quad \text{and} \quad \int_C f(x,y,z)\,ds$$

and the line integral of the vector field $\mathbf{F}$.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P(x,y)dx + Q(x,y)dy \quad \text{and} \quad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

Here are more details:

LINE INTEGRALS OF SCALAR FUNCTIONS
If $f(x,y)$ is a surface, then

$$\begin{aligned}
\int_C f(x,y)\,ds &= \int_a^b f(x(t),y(t))\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt &&= \text{‘line integral with respect to arc length’}\\
\int_C f(x,y)\,dx &= \int_a^b f(x(t),y(t))\,\tfrac{dx}{dt}\,dt &&= \text{‘line integral with respect to } x\text{’}\\
\int_C f(x,y)\,dy &= \int_a^b f(x(t),y(t))\,\tfrac{dy}{dt}\,dt &&= \text{‘line integral with respect to } y\text{’}
\end{aligned}$$

The last two above typically appear together in the form:

$$\int_C f(x,y)dx + f(x,y)dy = \int_C f(x,y)dx + \int_C f(x,y)dy = \int_a^b f(x(t),y(t))\,\frac{dx}{dt}\,dt + \int_a^b f(x(t),y(t))\,\frac{dy}{dt}\,dt$$

There are identical looking definitions on $\mathbb{R}^3$ for a function $f(x,y,z)$. The arc length then has $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$ and there is a line integral with respect to z. And you often see

$$\int_C f(x,y,z)dx + f(x,y,z)dy + f(x,y,z)dz = \int_C f(x,y,z)dx + \int_C f(x,y,z)dy + \int_C f(x,y,z)dz.$$

LINE INTEGRALS OF VECTOR FUNCTIONS
If $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a vector force field, then the work done my the field on a particle moving along the curve C given by $\mathbf{r}(t)$ is given by

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_C \mathbf{F}\cdot d\mathbf{r}\\
&= \int_C \langle P(x,y), Q(x,y)\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt\\
&= \int_C P(x,y)dx + Q(x,y)dy
\end{aligned}$$

A similar definition follows for $\mathbf{R}^3$.

PARAMETERIZING

We will learn various theorems that possibly simplify the calculation of these line integrals in special cases in section 16.3 and 16.4. But the direct way, and often still the fastest in many cases, for computing line integrals is to parameterize the curve. Sometimes you are given this parameterization and sometimes you have to find it. This often is the hardest part for students new to line integrals. Here are some tips for finding a parameterization:

1. METHOD 1: Simply let $x = t$ (or $y = t$) and solve for the other variable in terms of t. That is, if you curve is along $y = f(x)$ from (x_0, y_0) to (x_1, y_1). Then you let

   $$x = t, y = f(t) \text{ for } x_0 \leq t \leq x_1.$$

   Examples:

   - $y = x^3 + 2$ from $(0, 2)$ to $(2, 10)$. One parameterization is $x = t$, $y = t^3 + 2$ for $0 \leq t \leq 2$.
   - $x = \sqrt{(y+1)}$ from $(1, 0)$ to $(2, 3)$. One parameterization is $x = \sqrt{(t+1)}$, $y = t$ for $0 \leq t \leq 3$ (note, we picked $y = t$ for convenience).
   - $y = \sin(x) + x^2$ from $(0, 0)$ to (π, π^2). One parameterization is $x = t$, $y = \sin(t) + t^2$ for $0 \leq t \leq \pi$.
   - $y = x^3 + 2$ from $(2, 10)$ to $(0, 2)$. This the first example again with *opposite orientation*. In this situation you can either run the bounds in the opposite order $x = t$, $y = t^3 + 2$ from $t = 2$ to $t = 0$, or use $x = -t$, $y = (-t)^3 + 2$ for $0 \leq t \leq 2$.

2. METHOD 2: Consider the line segment from (x_0, y_0) to (x_1, y_1). We can quickly parameterize the line segment by

   $$x = (x_1 - x_0)t + x_0, y = (y_1 - y_0)t + y_0 \text{ for } 0 \leq t \leq 1.$$

   Examples:

   - Segment from $(1, 3)$ to $(-4, 7)$. One parameterization is $x = (-4 - 1)t + 1 = -5t + 1$, $y = (7 - 3)t + 3 = 4t + 3$ for $0 \leq t \leq 1$.
   - Segment from $(5, 2, 8)$ to $(-1, 4, 3)$. One parameterization is $x = (-1 - 5)t + 5 = -6t + 5$, $y = (4 - 2)t + 2 = 2t + 2$, $z = (3 - 8)t + 8 = -5t + 8$ for $0 \leq t \leq 1$.

3. METHOD 3: For circles and ellipses we can, and should, use trig as follows:
   Circles:

   $$x^2 + y^2 = r^2 \Rightarrow x = r\cos(t), y = r\sin(t) \text{ for } 0 \leq t \leq 2\pi \text{ (counterclockwise orientation)}$$

   Ellipses:

   $$\frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1 \Rightarrow x = a\cos(t), y = b\sin(t) \text{ for } 0 \leq t \leq 2\pi \text{ (counterclockwise orientation)}$$

   Examples:

   - $x^2 + y^2 = 9$, counterclockwise orientation in first quadrant.
     One parameterization: $x = 3\cos(t)$, $y = 3\sin(t)$ for $0 \leq t \leq \frac{\pi}{2}$.
   - $\frac{1}{4}x^2 + 9y^2 = 1$ counterclockwise orientation only lower half ($y \leq 0$).
     One parameterization: $x = 2\cos(t)$, $y = \frac{1}{3}\sin(t)$ for $\pi/2 \leq t \leq 2\pi$.

ORIENTATION

As we learned in class, a parameterization gives a particular orientation (the direction forward in the parameter value). If C is a curve with a particular orientation, we say that $-C$ is the curve with opposite orientation. Here are some notes on orientation:

1. We discovered that orientation doesn't matter for line integrals over scalar functions with respect to arc length. That is orientation doesn't matter for $\displaystyle\int_C f(x,y)\,ds$ and $\displaystyle\int_C f(x,y,z)\,ds$. So in this case $\displaystyle\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds$.
2. But orientation does matter for all the other integrals of this section, and it always give the opposite sign ($\int_{-C} = -\int_C$). Most importantly, it matters for $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$, so $\displaystyle\int_{-C} P\,dx + Q\,dy + R\,dz = -\int_C P\,dx + Q\,dy + R\,dz$.
3. *What does this mean for you?* When you give a parameterization, you must check that the orientation matches the requested orientation. You can quickly do this by plugging in the endpoints of your parameterization and making sure you start and end at the correct points. AND if you have the opposite orientation from what is desired, there is a quick fix. Integrate over your parameterization with a negative in front of the integral!