Final answers and sketches of some solutions for the sample final.
CORRECTIONS made in limits in #3 and last line of answer to #6.

1. (a) Scalar, vector, nonsense, nonsense, vector, scalar. (Note that the last one is the length of a vector, not a vector.)

   (b) Only $curl(grad(f))$ vanishes no matter what the fields are.

2. *Many choices of the parametrizations are possible, but the final answers should be the same for any choice.*

   Let C_1 be the straight section and C_2 the curved section of C. Parametrize C_1: $\langle t, 0\rangle$, $0 \le t \le 2$. For C_2, $\langle 2cos(t), 2sin(t)\rangle$, $-\pi/2 \le t \le 0$.

   (a) $\int_{C_1} x\,ds = \int_0^2 t\,dt = ... = 2$. On C_2, $ds = 2dt$ and $\int_{C_2} x\,ds = \int_{-\pi/2}^0 2\cos(t)\,2dt = ... = 4$. So $\int_C x\,ds = \int_{C_1} x\,ds + \int_{C_2} x\,ds = 6$.

   (b) $dy = 0$ on C_1, so $\int_{C_1} x\,dy = 0$ and therefore

$$\int_C x\,dy = \int_{C_2} x\,dy = \int_0^{-\pi/2} 2\cos(t)\frac{d}{dt}(2\sin(t))\,dt = ... = -\pi$$

3. (a) $\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\int_{-5}^{-\sqrt{x^2+y^2}} (e^x + z^2)\,dz\,dy\,dx$

   (b) $\int_0^{2\pi}\int_0^5\int_{-5}^{-r} (e^{r\cos(\theta)} + z^2) r\,dz\,dr\,d\theta$ or
   $\int_0^{2\pi}\int_{3\pi/4}^{\pi}\int_0^{-5/\cos(\varphi)} (e^{\rho\sin(\varphi)\cos(\theta)} + \rho^2\cos^2(\varphi)))\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$

4. *Many choices of the parametrizations are possible.*

   (a) $\mathbf{r}(u, v) = \langle 4\cos(v), u, 2\sin(v)\rangle$ for $0 \le u \le 10$ and $0 \le v \le 2\pi$. Then $\mathbf{r}_u \times \mathbf{r}_v = \langle 2\cos(v), 0, 4\sin(v)\rangle$, which points outward and so this parametrization gives the same orientation as the one specified.

   (b) For C_1, parametrize by $\langle 4\cos(t), 0, 2\sin(t)\rangle$ with $0 \le t \le 2\pi$. For C_2, just change y from 0 to 10: $\langle 4\cos(t), 10, 2\sin(t)\rangle$ with $0 \le t \le 2\pi$. For both of these, the curve starts at a point in the xy-plane where $x > 0$ and moves upward (increasing z). For C_1, the surface is on the right when going in this direction, so the parametrization gives the opposite orientation to the one consistent with the specified one on S. For C_2, the surface is on the left when going in this direction, so this orientation for C_2 is consistent with the specified one on S.

5. Choosing a simple parametrization, $\mathbf{r}(u, v) = \langle 1 - u^2 - v^2, u, v\rangle$, with $u^2 + v^2 \le 1$ and $u \le v$ (an eighth of a unit disk). Then $\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 2u, 2v\rangle$, which points away from the origin. So I'll have to put a minus sign in front of the integral.

$$-\int_0^{1/\sqrt{2}}\int_u^{\sqrt{1-u^2}} \langle 1 - u^2 - v^2, u, v\rangle \cdot \langle 1, 2u, 2v\rangle\,dv\,du = -\int_0^{1/\sqrt{2}}\int_u^{\sqrt{1-u^2}} (1 + u^2)\,du\,dv.$$

   Change to polar coordinates to get simpler limits of integration and evaluate to get $-3\pi/16$.

   By using polar coordinate ideas in the parametrization, you can get the simpler limits more directly: $\mathbf{r}(u, v) = \langle 1 - u^2, u\cos(v), u\sin(v)\rangle$, with $0 \le u \le 1$ and $\pi/4 \le v \le \pi/2$. The orientation is still opposite to the given one. The integral becomes

$$-\int_{\pi/4}^{\pi/2}\int_0^1 (u + u^3)\,du\,dv$$

   (with of course the same final answer of $-3\pi/16$).

6. Computing directly, you should get 9/2, -17/2, and 1 for the integrals along the three line segments, for a total integral around C of -3.

   OR, notice, in addition to the the hint that you may use a theorem, that the curl of $\mathbf{F}$ will be a constant vector field, so Stokes's Theorem seems like a good idea. Compute $curl(\mathbf{F}) = 5\mathbf{i}-3\mathbf{j}+7\mathbf{k}$. Fill in C with a planar surface T: it's a triangle with unit normal $\mathbf{n} = (\mathbf{i}-\mathbf{k})/\sqrt{2}$ giving the compatible orientation. You can parametrize and integrate as usual, but $\mathbf{F}\cdot\mathbf{n} = (5-7)/\sqrt{2} = -\sqrt{2}$, a constant. So we can compute the integral by multiplying this constant times the area of T. The perpendicular sides of the triangle have lengths 3 and $\sqrt{2}$, so altogether we get $\iint_T \mathbf{F}\cdot\mathbf{n}dA = -\sqrt{2}(3\sqrt{2}/2) = -3$.

7. (a) The boundary of E is the union of S_1 and S_2 together, but with the reversed orientation on S_2 (because "downward" is "outward" on the bottom). Therefore the Divergence Theorem tells us that

$$\iiint_E div(\mathbf{F})dV = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} - \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$$

   (where the integrals over S_1 and S_2 mean with the original orientations of those surfaces). Adding the integral over S_2 to both sides gives us the desired result.

   (b) Use part (a). Compute $div(\mathbf{F}) = 2$, so $\iiint_E div(\mathbf{F})dV = 2$(volume of hemisphere of radius 3) $= 36\pi$ (using remembered formula for the volume of a sphere, or computing using spherical coordinates). On S_2, the unit normal is $\mathbf{k}$, so we only need the $\mathbf{k}$-component of $\mathbf{F}$, which is $x+z = x$ on S_2. By symmetry (or calculation), $\iint_{S_2} x\,dx\,dy = 0$. So $\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = 36\pi + 0 = 36\pi$.

Final answers and sketches of some solutions for the additional review problems:

1. (a) Compute the curl of each vector field. For $\mathbf{F}$, it's the zero vector field. As $\mathbf{F}$ and the partials of its components are defined on all of space, we know that $\mathbf{F}$ is conservative. A potential function is given by $f(x,y,z) = \dfrac{x^2}{2} + y + xz$. (It's a good idea to check by computing $grad(f)$ and make sure it gives you $\mathbf{F}$ back again.)

   The curl $\mathbf{G} = \langle 0,0,-2\rangle$ is not zero, so $\mathbf{G}$ is not conservative. (If you made a sign error and thought the curl was zero, then you should discover your error when you try to find the potential function: you won't be able to solve the equations, or if you think you do, when you compute the gradient you won't get $\mathbf{G}$.)

   (b) $\int_C \mathbf{F}\cdot d\mathbf{r} = f(4,2,20) - f(0,0,0) = ... = 90$
   (Or, compute using the parametrization below.)

   To get a parametrization of C, notice that the first equation gives z in terms of x and y, and the second gives x in terms of y. So let $\mathbf{r}(t) = \langle 2t, t, 5t^2\rangle$, with $0 \le t \le 2$. Then compute $\int_C \mathbf{G}\cdot d\mathbf{r} = ... = e^{20} - 1$.

2. (a) $\nabla g = \langle z^2, 1, 2xz\rangle$, so $\nabla g(\mathbf{r}(5)) = \langle 9,1,12\rangle$.
   $h'(5) = \nabla g(\mathbf{r}(5))\cdot\mathbf{r}'(5) = \langle 9,1,12\rangle\cdot\langle -1,\pi,2\rangle = 15+\pi$

   (b) $9x + y + 12z = 47$

   (c) We want $\mathbf{u} = \langle a,b,c\rangle$ so that $0 = \nabla g(2,-7,3)\cdot\mathbf{u} = 9a+b+12c$. One answer is $\mathbf{u} = \dfrac{\langle 1,3,-1\rangle}{\sqrt{11}}$. (Technically, "direction" means unit vector; one may also say "in the direction of $\langle 1,3,-1\rangle$." But I'm not very concerned about this technicality.)

3. (a) $\mathbf{r}(u, v) = \langle u, v^2, v\rangle$ with $u^2 + v^2 \le 4$, is the easiest one to write down.

   $\mathbf{r}(u, v) = \langle u\cos(v), u^2\sin^2(v), u\sin(v)\rangle$ with $0 \le u \le 2$, $0 \le v \le 2\pi$ also works.

   For both of these, $\mathbf{r}_u \times \mathbf{r}_v$ has negative $\mathbf{j}$ component, so the orientation give by the parametrization is opposite to the specified orientation.

   (b) I'll use $\mathbf{q}(t)$ for the curve, because we're already using $\mathbf{r}$ for the surface:
   $\mathbf{q}(t) = \langle 2\cos(t), 4\sin^2(t), 2\sin(t)\rangle$, for $0 \le t \le 2\pi$.

   The orientation consistent with the given orientation of S is counterclockwise as viewed from the positive y-axis, which is opposite to the orientation of the parametrization $\mathbf{q}$.

   (c) Using Stokes' Theorem,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = -\int_0^{2\pi} \mathbf{F}(\mathbf{q}(t)) \cdot \mathbf{q}'(t)dt = ... = 8\pi.$$

   (The minus sign in front of the integral comes from orientation discussion in part (b).)

4. (a) $f(x, y, z) = 2z$ is constant around each horizontal circular cross-section. The length of such a cross-section is 6π, so the surface integral is equal to $6\pi \int_0^5 2z dz = 150\pi$.

   (b) doesn't make sense, because we can't take a dot product of a scalar function with the vector $d\mathbf{S} = \mathbf{n}dS$.

   (c) This integral is the flux of $\mathbf{F}$ through S. But $\mathbf{F}$ is constant, so the flux in one side of S is equal to the flux out the other side. Thus the integral is zero.

   Same idea, described slightly differently: By symmetry of S, we have $\mathbf{F} \cdot \mathbf{n}$ with the same magnitude and opposite sign at the points (a, b, c) and $(-a, -b, c)$ in S, so the integral will be zero.

   I didn't ask you about the fourth variation, $\iint_S \mathbf{F} dS$. This expression "makes sense," its meaning is to compute the integral of each of the components of $\mathbf{F}$, resulting in three numbers that are the components of a single vector. But it's not a type of calculation we've done, because it's not a concept that is particularly useful.