MATH 324A (Autumn 2009)
Final Exam

Student name: ____________________________________

Student number: __________________________________

Signature: ______________________________________

Do not start working until instructed to do so.

You have 110 minutes.

Please show your work.

Scientific, but not graphing calculators are allowed.

You may use one 8.5 by 11 double-sided sheet of handwritten notes.

Problem 1 (30 points)	
Problem 2 (10 points)	
Problem 3 (10 points)	
Problem 4 (10 points)	
Problem 5 (20 points)	
Problem 6 (20 points)	
Total	

Problem 1 (30 points) Evaluate the following integrals.

(a) $I = \int_C y dx - x^2 dy$, where C is the curve from $(0, 0)$ to $(1, 1)$ along the parabola $y = x^2$.

(b) $I = \int_S F \cdot dS$, where $F(x, y, z) = (1, 1, 1)$ and $S \subseteq \mathbb{R}^3$ is the surface parametrized by $\Gamma(u, v) = (u^2, v^2, u + v)$, $0 \leq u \leq 1$, $0 \leq v \leq 1$.

(c) $I = \int_S 1 + x^3y^2\,dS$, where S is the surface $z = x^2 + y^2$ above the closed unit disk $D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\}$.

(d) $I = \int_{-1}^{1} \int_{|x|-1}^{1-|x|} (x + y)^2\,dy\,dx$. Hint: change of variables $u = x + y$ and $v = x - y$.

(e) $I = \int_C y\,ds$, where $C = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1, y \geq 0\}$ is the upper half of the unit circle.

(f) $I = \int_E z\,dV$, where $E = \{(x, y, z) \in \mathbb{R}^3 \colon \sqrt{x^2 + y^2} \leq z \leq 2\}$.

Problem 2 (10 points) Consider the vector field $F(x, y) = (2xy, x^2 + 3y^2)$. It is given that F is conservative on $\mathbb{R}^2$.

(a) Find ALL functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ for which $F = \nabla f$.

(b) Evaluate $\int_C 2xydx + (x^2 + 3y^2)dy$, where C is a curve from $(1, 2)$ to $(2, 8)$ along the parabola $y = 2x^2$.

Problem 3 (10 points) Let $E \subseteq \mathbb{R}^3$ be the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where a, b, $c > 0$ are constants.

(a) Using spherical coordinates, write down a parametrization of E. Hint: E is the image of the unit sphere under the linear map $(x, y, z) \mapsto (ax, by, cz)$.

(b) Setup an iterated integral of the form $\int_a^b \int_c^d f(\phi, \theta)\, d\phi\, d\theta$ which represents the area of E. You don't need to simplify the expression of $f(\phi, \theta)$ or evaluate the integral. Your answer may contain a cross product.

Problem 4 (10 points) Let $F(x,y) = (y^2, x^2)$.

(a) Is F conservative on $\mathbb{R}^2$? Explain.

(b) Evaluate the curve integral $\int_\gamma F \cdot ds$, where

$$\gamma(t) = \left(t^{t^2+1} \sin\left(\frac{\pi}{6} t^{t^2+1}\right), t^{2(t^2+1)} \sin^2\left(\frac{\pi}{6} t^{t^2+1}\right)\right)$$

is defined for $0 \leq t \leq 1$. Hint: let $u = t^{t^2+1} \sin\left(\frac{\pi}{6} t^{t^2+1}\right)$ and re-parametrize the curve by setting $\gamma_1(u) = (u, u^2) = \gamma(t)$. You may need to figure out the range of u.

Problem 5 (20 points) Let $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1, z \geq 0\}$ be the upper half of the unit sphere, with outward orientation. That is, the outward unit normal vector field

$$n(x, y, z) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$$

is chosen for computing surface integral. Let $C = \{(x, y, 0) \in \mathbb{R}^3 \colon x^2 + y^2 = 1\}$ be oriented counter-clockwise when viewed from above. Let $D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\}$ be the closed unit disk in xy-plane.

(a) Use Stokes' theorem and Green's theorem to show that, for any smooth vector field $F = (P, Q, R)$ defined on $\mathbb{R}^3$,

$$\int_S \operatorname{curl}(F) \cdot dS = \int_D \frac{\partial Q}{\partial x}(x, y, 0) - \frac{\partial P}{\partial y}(x, y, 0)\, dA.$$

(b) Evaluate the flux integral $\int_S \operatorname{curl}(F) \cdot dS$, where $F(x, y, z) = (x^2y^3, -2xyz, 3z^2)$.

Problem 6 (20 points) Find the volume of the solid $E = \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 \leq 1, z \geq \frac{1}{2}\right\}$.