Problem 1 (20 points) Evaluate the following integrals.

(a) $I = \int_D e^{x^2+y^2}\,dA$, where $D = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 3\}$.

$$I = \int_0^{2\pi}\int_0^{\sqrt{3}} e^{r^2}\, r\, dr\, d\theta$$

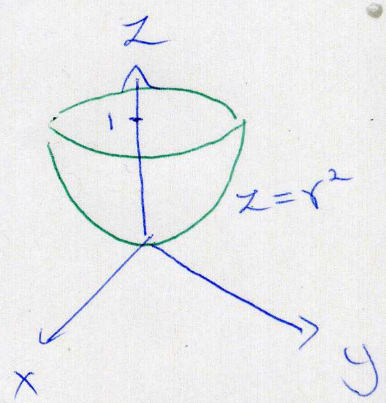
$$= 2\pi \int_0^{\sqrt{3}} e^{r^2}\, r\, dr$$

$$= 2\pi \int_0^{3} e^{u} \frac{du}{2}$$

$$= \pi(e^3 - 1).$$

(b) $I = \int_E x + y + z\,dV$, where $E = \{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 \leq 1,\ x^2 + y^2 \leq z \leq 1\}$. (Think before you compute.)

Note: E is symmetric in yz-plane and xz-plane



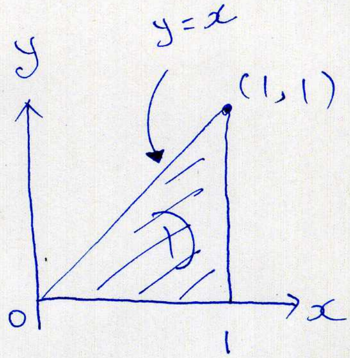
$\Rightarrow \int_E x\,dV = \int_E y\,dV = 0$

So, $I = \int_E z\,dV$

$$= \int_0^{2\pi}\int_0^1\int_{r^2}^1 z\, r\, dz\, dr\, d\theta$$

$$= \frac{\pi}{3}.$$

(c) $I = \int_D e^{x^2}\, dA$, where $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1,\ y \leq x \leq 1\}$.



Switch the order of dx, dy

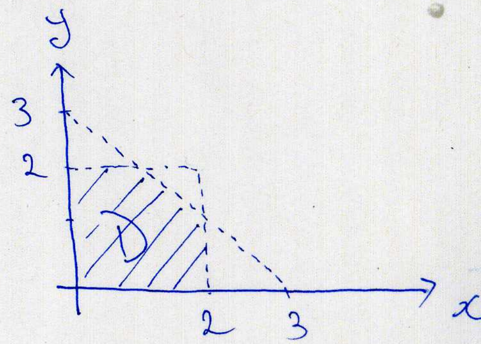
$$I = \int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

$$= \int_0^1 e^{x^2}\, x\, dx$$

$$= \int_0^1 e^u \frac{du}{2} \quad \begin{pmatrix} u = x^2 \\ du = 2x\,dx \end{pmatrix}$$

$$= \frac{1}{2}(e - 1).$$

(d) $I = \int_D x - y + 5\, dA$, where $D = \{(x, y) \in [0, 2] \times [0, 2] : x + y \leq 3\}$. (You may simplify the computation with the fact that the region D has a line of symmetry.)



Note: D is symmetric in the line $y = x$

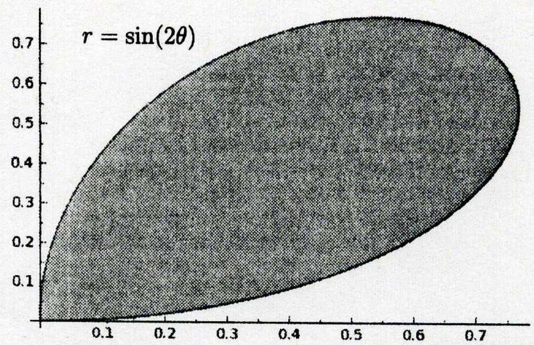
$$\Rightarrow \int_D x - y\, dA = 0$$

Therefore,

$$I = \int_D 5\, dA = 5\ \text{area}(D)$$

$$= \frac{35}{2}.$$

Problem 2 (10 points) Find the area enclosed by the curve $r = \sin(2\theta)$, $0 \leq \theta \leq \frac{\pi}{2}$.



Jacobian

$$\text{area} = \int_0^{\frac{\pi}{2}} \int_0^{\sin(2\theta)} r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2(2\theta)}{2} \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1 - \cos(4\theta)}{4} \, d\theta$$

$$= \frac{\pi}{8}.$$

Recall

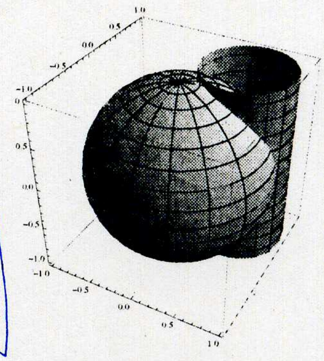
$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

(double angle formula)

Problem 3 (10 points) Consider the solid that the cylinder $r = \cos\theta$ cuts out of the unit sphere $x^2 + y^2 + z^2 = 1$.

(a) Setup a triple integral which represents the volume of the solid.

$$\text{Vol} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos(\theta)} \int_{-\sqrt{1-r^2}}^{+\sqrt{1-r^2}} r \, dz \, dr \, d\theta$$



$$\left[\text{or} = 4 \int_0^{\frac{\pi}{2}} \int_0^{\cos(\theta)} \int_0^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta \right]$$

(b) Compute the volume of the solid.

$$\text{Vol} = 4 \int_0^{\frac{\pi}{2}} \int_0^{\cos(\theta)} r\sqrt{1-r^2} \, dr \, d\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} \int_{\sin^2(\theta)}^{1} \sqrt{u} \, \frac{du}{2} \, d\theta \quad \left(\begin{array}{l} u = 1 - r^2 \\ du = -2r\,dr \end{array} \right)$$

$$= \frac{4}{3} \int_0^{\frac{\pi}{2}} 1 - \sin^3(\theta) \, d\theta$$

$$= \frac{4}{3} \left[\theta + \cos(\theta) - \frac{\cos^3(\theta)}{3} \right]_{\theta=0}^{\frac{\pi}{2}}$$

$$= \frac{2}{9}(3\pi - 4).$$

Problem 4 (10 points) Let (X, Y, Z) be a uniformly distributed random point on the unit sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Let (Θ, Φ) be the spherical coordinates of the point, given by

$$\begin{cases} X = \sin(\Phi)\cos(\Theta) \\ Y = \sin(\Phi)\sin(\Theta) \\ Z = \cos(\Phi) \end{cases}$$

You are told that the probability joint density function of (Θ, Φ) is

$$f(\theta, \phi) = \begin{cases} \frac{\sin(\phi)}{4\pi}, & (\theta, \phi) \in [0, 2\pi] \times [0, \pi] \\ 0, & \text{otherwise} \end{cases}$$

What is the probability that $|X| \le \frac{1}{2}$? (Hint: the sphere $\mathbb{S}^2$ is invariant under rotation.)

$$P(|X| \le \tfrac{1}{2}) = P(|Z| \le \tfrac{1}{2}) \quad \text{by symmetry}$$

$$= P(-\tfrac{1}{2} \le \cos(\Phi) \le \tfrac{1}{2})$$

$$= \int_0^{2\pi} \int_{\cos^{-1}(\frac{1}{2})}^{\cos^{-1}(-\frac{1}{2})} \frac{\sin(\phi)}{4\pi} \, d\phi \, d\theta$$

$$= \frac{1}{2} \,_{\text{//}}$$

<u>Remark</u> We switch from X to Z because $|X| \le \frac{1}{2}$ corresponds to a more complicated region in $\theta\phi$-plane.