Problem 1 (20 points) Evaluate the following integrals.

(a) $I = \int_D e^{x^2+y^2}\, dA$, where $D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 2\}$.

$$I = \int_0^{2\pi}\int_0^{\sqrt{2}} e^{r^2} r\, dr\, d\theta \quad \text{(polar coordinates)}$$

$$= 2\pi \int_0^{\sqrt{2}} e^{r^2} r\, dr$$

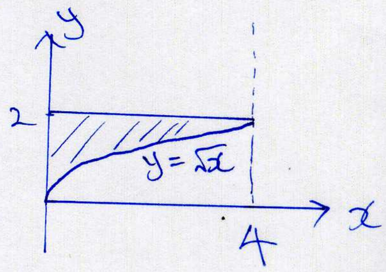
$$= 2\pi \int_0^{2} e^{u} \frac{du}{2} \quad \left(\begin{array}{c} u = r^2 \\ du = 2r\, dr \end{array}\right)$$

$$= \pi\left(e^2 - 1\right) \quad //$$

(b) $I = \int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\, dy\, dx$.

Switching order of integration gives

$$I = \int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\, dx\, dy$$



$$= \int_0^2 \frac{y^2}{y^3+1}\, dy$$

$$= \int_1^9 \frac{du}{3u} \quad \left(\begin{array}{c} u = y^3 + 1 \\ du = 3y^2\, dy \end{array}\right)$$

$$= \frac{1}{3}\log(9) \quad \text{or} \quad \frac{2}{3}\log(3)$$

natural log

(c) $I = \int_E x^2 + y^2 + z^2 \, dV$, where $E = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 \leq 1, z \geq 0\}$.

Using spherical coordinates,

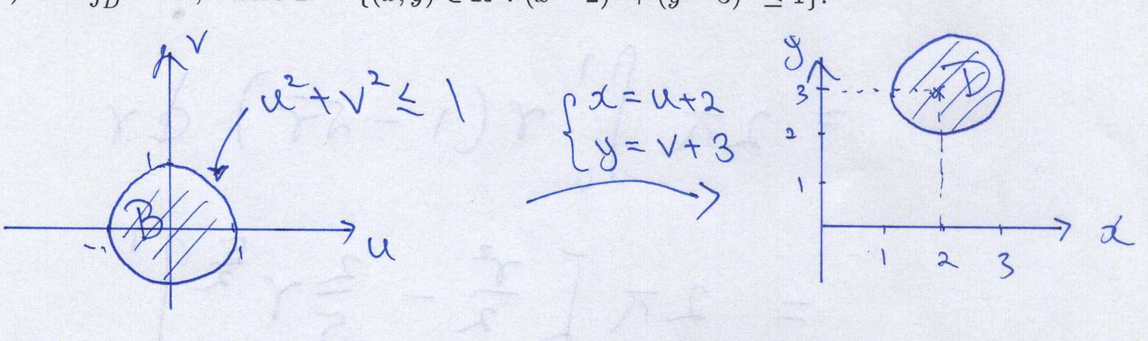
$$I = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^2 \cdot \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \left(\int_0^{\frac{\pi}{2}} \sin(\phi) \, d\phi \right) \left(\int_0^1 \rho^4 \, d\rho \right)$$

$$= 2\pi (1) \left(\frac{1}{5} \right)$$

$$= \frac{2\pi}{5} \text{ ,,}$$

(d) $I = \int_D x^2 \, dA$, where $D = \{(x, y) \in \mathbb{R}^2 \colon (x - 2)^2 + (y - 3)^2 \leq 1\}$.



By a translation, Jacobian

$$I = \int_B (u+2)^2 \cdot 1 \, dA(u,v) \qquad (\because \text{ odd function})$$

$$= \int_B u^2 + 4u + 4 \, dA(u,v)$$

$$= \int_B u^2 \, dA(u,v) + 0 + 4 \, \text{area}(B)$$

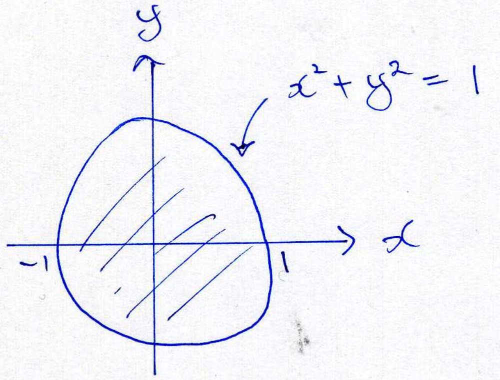
$$= \int_0^{2\pi} \int_0^1 r^2 \cos^2(\theta) \cdot r \, dr \, d\theta + 4\pi$$

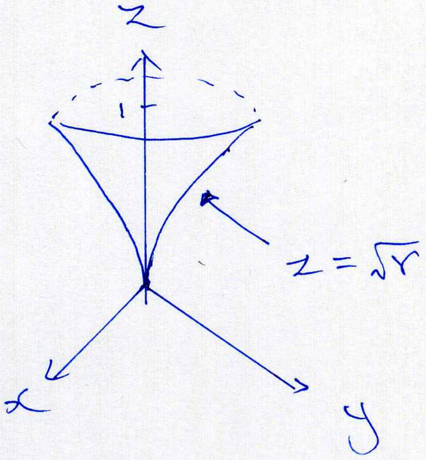
$$= \left(\int_0^{2\pi} \cos^2(\theta) \, d\theta \right) \left(\int_0^1 r^3 \, dr \right) + 4\pi$$

$$= \left(\int_0^{2\pi} \frac{1 + \cos(2\theta)}{2} \, d\theta \right) \left(\frac{1}{4} \right) + 4\pi = \frac{17\pi}{4}$$

Problem 2 (10 points) Find the volume of the solid

$$E = \left\{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2)^{\frac{1}{4}} \le z \le 1\right\}.$$





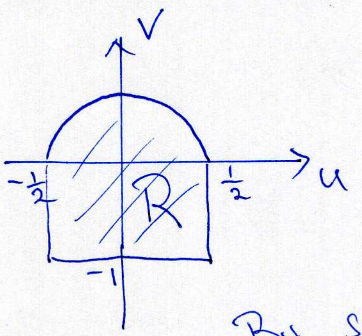
$$\mathrm{Vol}(E) = \int_0^{2\pi}\int_0^1\int_{\sqrt{r}}^1 r\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^1 r(1-\sqrt{r})\,dr$$

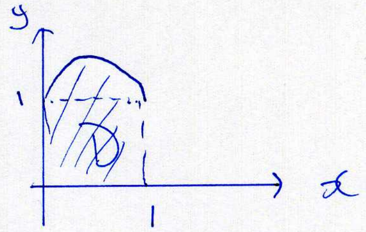
$$= 2\pi \left[\frac{r^2}{2} - \frac{2}{5} r^{\frac{5}{2}}\right]_{r=0}^{1}$$

$$= \frac{\pi}{5}$$

Problem 3 (10 points) Find the center of mass of a lamina which has constant density $\rho(x,y) \equiv 1$ and occupies the region $D = [0,1]\times[0,1] \cup \{(x,y)\in\mathbb{R}^2\colon (x-\frac{1}{2})^2 + (y-1)^2 \le \frac{1}{4}\}$. (The region is the union of a unit square and a semi-disk.)



$$\begin{cases} x = u + \frac{1}{2} \\ y = v + 1 \end{cases} \longrightarrow$$



By symmetry, $\bar{u} = 0$.

Total mass:

$$m = \int_R 1\, dA(u,v) = \text{area}(R) = 1 + \frac{\pi}{8}$$

Find $\bar{v}$:

$$m\bar{v} = \int_R v\, dA(u,v)$$

$$= \left(\int_{\square} + \int_{\frown}\right) v\, dA(u,v)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{0} v\, dv\, du + \int_0^{\pi}\int_0^{\frac{1}{2}} r\sin(\theta)\cdot r\, dr\, d\theta$$

$$= \left[\frac{v^2}{2}\right]_{v=-1}^{0} + \left(\int_0^{\pi}\sin(\theta)\,d\theta\right)\left(\int_0^{\frac{1}{2}} r^2\,dr\right)$$

$$= -\frac{1}{2} + (2)\left(\frac{1}{24}\right)$$

$$= -\frac{5}{12}$$

$$\Rightarrow \bar{v} = -\frac{5}{12(1+\frac{\pi}{8})}$$

Therefore,

$$(\bar{x},\bar{y}) = (\bar{u},\bar{v}) + \left(\tfrac{1}{2}, 1\right)$$

$$= \left(\frac{1}{2},\ 1 - \frac{5}{12(1+\frac{\pi}{8})}\right)$$

Problem 4 (10 points) Consider the change of variables

$$\begin{cases} x = u + 2v \\ y = u - 2v \end{cases}$$

(a) Verify that the image of the unit circle $u^2 + v^2 = 1$ under the above transformation is the ellipse $5x^2 + 6xy + 5y^2 - 16 = 0$. (You don't need to explain why it is an ellipse.)

Solving for u, v gives $u = \frac{x+y}{2}$ and $v = \frac{x-y}{4}$.
The image of $u^2+v^2=1$ has equation

$$\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{4}\right)^2 = 1$$

$$\frac{x^2+2xy+y^2}{4} + \frac{x^2-2xy+y^2}{16} = 1$$

which simplifies to

$$5x^2+6xy+5y^2-16=0$$

(b) Evaluate the double integral $I = \int_R \sqrt{4(x+y)^2 + (x-y)^2}\, dA$, where R is the region bounded by the ellipse in the previous part.

Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -4$$

So,

$$I = \int_{\{u^2+v^2\le 1\}} \sqrt{4(2u)^2+(4v)^2} \cdot 4\, du\, dv$$

$$= 16 \int_{\{u^2+v^2\le 1\}} \sqrt{u^2+v^2}\, du\, dv$$

$$= 16 \int_0^{2\pi}\int_0^1 r \cdot r\, dr\, d\theta \quad \text{(polar coordinates)}$$

$$= \frac{32\pi}{3}$$