Problem 1 (30 points) Evaluate the following integrals.

(a) $I = \int_C F \cdot ds$, where $F(x,y) = (y, -x)$ and C is the unit circle $x^2 + y^2 = 1$ oriented counter-clockwise.

parametrization of C: $\gamma(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

$\gamma'(t) = (-\sin t, \cos t)$

$F(\gamma(t)) = (\sin t, -\cos t)$

$$\int_C F \cdot ds = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{2\pi} -1\, dt = -2\pi$$

<u>OR</u> use Green's theorem,

$$\int_C F \cdot ds = \int_D \frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\, dA$$

$$= \int_D -2\, dA = -2\,\text{area}(D) = -2\pi,$$

where $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

(b) $I = \int_D xy\, dA$, where $D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le x^2\}$.

$$I = \int_0^1 \int_0^{x^2} xy\, dy\, dx$$

$$= \int_0^1 \left[\frac{xy^2}{2} \right]_{y=0}^{x^2} dx$$

$$= \int_0^1 \frac{x^5}{2}\, dx$$

$$= \frac{1}{12}.$$

(c) $I = \int_S F \cdot dS$, where $F(x,y,z) = (x^2, y^2 z, -1)$ and S is the part of the surface $z = x^2 + y^2$ which lies inside the cylinder $x^2 + y^2 = 1$. Use the upward orientation of S.

Write $F = (P, Q, R)$. Let $g(x,y) = x^2 + y^2$ and $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

$$I = \int_D -P g_x - Q g_y + R \, dA$$

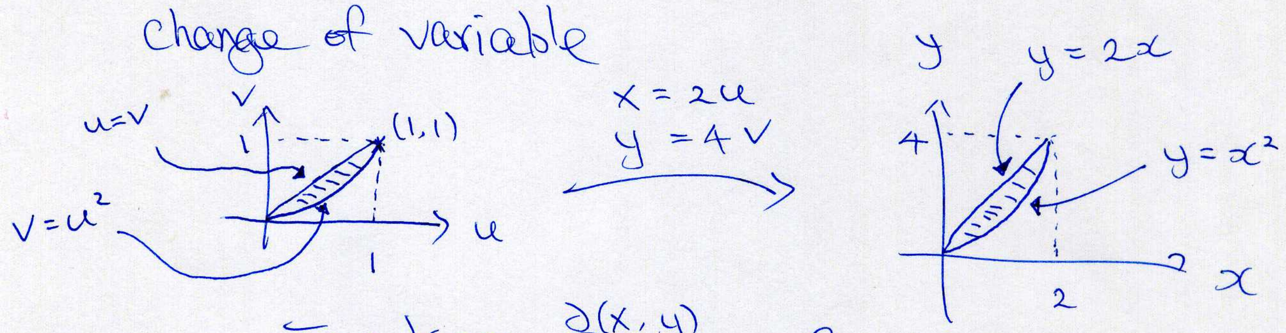
$$= \int_D -2x^3 - y^2(x^2+y^2)\, 2y - 1 \, dA$$

$$= 0 + 0 - \text{area}(D) \quad (\because \text{odd function}$$

$$= -\pi \,_{\text{"}}$$

(d) $I = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} x^2 + y^2 \, dx dy$.

Change of variable



Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = 8$

$$I = \int_0^1 \int_{u^2}^{u} (4u^2 + 16v^2)\, 8 \, dv du$$

$$= 32 \int_0^1 \int_{u^2}^{u} u^2 + 4v^2 \, dv du$$

$$= 32 \int_0^1 u^2(u - u^2) + \frac{4}{3}(u^3 - u^6) \, du$$

$$= \frac{216}{35} \,_{\text{"}}$$

(e) $I = \int_L x^2\, ds$, where L is the line segment from $(1,2,3)$ to $(4,5,6)$.

parametrization of L:

$$\gamma(t) = (1+3t,\ 2+3t,\ 3+3t),\quad 0 \le t \le 1$$

$$\gamma'(t) = (3,\ 3,\ 3)$$

$$|\gamma'(t)| = 3\sqrt{3}$$

$$I = \int_0^1 (1+3t)^2\, 3\sqrt{3}\, dt$$

$$= 3\sqrt{3} \int_0^1 1 + 6t + 9t^2\, dt$$

$$= 21\sqrt{3}$$ //

(f) $I = \int_S y\, dS$, where S is the surface $z = 3x + y^4$ above the region $D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le 1\}$.

$$I = \int_D y \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \int_D y \sqrt{10 + 16y^6}\, dA$$

$$= \int_0^1 y \sqrt{10 + 16y^6}\, dy$$

$$= \int_0^1 \sqrt{10 + 16u^3}\, \frac{du}{2} \qquad (u = y^2)$$

$$= \frac{\sqrt{2}}{2} \int_0^1 \sqrt{5 + 8u^3}\, du$$

Problem 2 (10 points) Let E be the ellipse $\frac{x^2}{a^a} + \frac{y^2}{b^2} = 1$, where a, $b > 0$ are constants.

(a) Write down a parametrization of E.

$$\gamma(t) = (a\cos t, b\sin t)$$
$$0 \leq t \leq 2\pi$$

(b) Use Green's theorem to show that the area bounded by E is πab. (Represent the area as a curve integral.)

$$\text{area} = \frac{1}{2}\int_\gamma x\,dy - y\,dx$$
$$= \frac{1}{2}\int_0^{2\pi} a\cos(t)\, b\cos(t) - b\sin(t)[-a\sin(t)]\, dt$$
$$= \frac{ab}{2}\int_0^{2\pi} dt$$
$$= \pi ab \,_{\shortparallel}$$

Problem 3 (10 points) Consider the vector field $F(x,y) = \big((y+2)\sin(2x), \sin^2(x)\big)$. It is given that F is conservative on $\mathbb{R}^2$.

(a) Find ALL functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ for which $F = \nabla f$.

$$\begin{cases} f_x(x,y) = (y+2)\sin(2x) & \text{——— } ① \\ f_y(x,y) = \sin^2(x) & \text{——— } ② \end{cases}$$

Integrate ② w.r.t. y:

$$f(x,y) = y\sin^2(x) + g(x) \quad \text{——— } ③$$

for some $g(x)$. Differentiate ③ w.r.t. x:

$$f_x(x,y) = 2y\sin(x)\cos(x) + g'(x)$$
$$= y\sin(2x) + g'(x) \quad \text{——— } ④$$

By ① and ④,

$$g'(x) = 2\sin(2x) \Rightarrow g(x) = -\cos(2x) + C$$

where C is a constant. Hence,

$$f(x,y) = y\sin^2(x) - \cos(2x) + C$$

(b) Evaluate $\int_\gamma F \cdot ds$, where $\gamma(t) = (te^t, t^3)$, $0 \le t \le 1$.

By Fundamental theorem of line integral,

$$\int_\gamma F \cdot ds = f(\gamma(1)) - f(\gamma(0))$$
$$= f(e,1) - f(0,0)$$
$$= \sin^2(e) - \cos(2e) + 1$$

Problem 4 (10 points) Let $F(x, y, z) = (x^2y, x^3, y)$ and let C be the intersection of the hyperbolic paraboloid $z = y^2 - x$ and the cylinder $x^2 + y^2 = 1$. Use Stokes' theorem to evaluate the integral $\int_C F \cdot ds$, where C is oriented as counter-clockwise viewed from above.

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2y & x^3 & y \end{vmatrix}$$

$$= (1, 0, 2x^2)$$

Let $P = 1$, $Q = 0$, $R = 2x^2$,

$g(x,y) = y^2 - x$, $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

By Stokes' theorem,

$$\int_C F \cdot ds = \int_S (1, 0, 2x^2) \cdot dS$$

$$= \int_D -Pg_x - Qg_y + R \, dA$$

$$= \int_D 1 + 2x^2 \, dA$$

$$= \text{area}(D) + 2\int_0^{2\pi}\int_0^1 r^2\cos^2(\theta)\, r \, dr\, d\theta$$

$$= \pi + 2\left(\int_0^{2\pi} \cos^2(\theta)\, d\theta\right)\left(\int_0^1 r^3 \, dr\right)$$

$$= \pi + \frac{1}{4}\int_0^{2\pi} 1 + \cos(2\theta)\, d\theta$$

$$= \pi + \frac{1}{4}(2\pi + 0) = \frac{3}{2}\pi \,\,''$$

''

Problem 5 (20 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Show that $\operatorname{curl} \nabla f = 0$. Hint: write $\nabla f = (f_x, f_y, f_z)$ and compute $\operatorname{curl} \nabla f$ from definition.

$$\operatorname{curl}(\nabla f) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix}$$

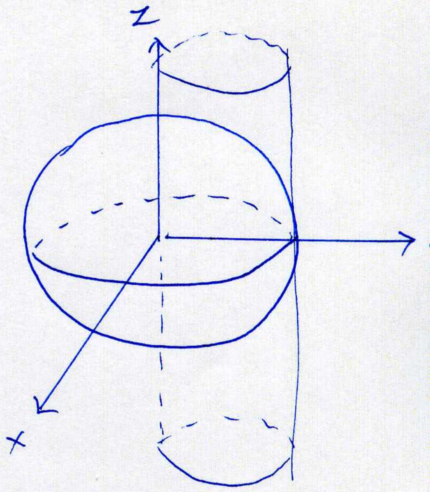
$$= (f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy})$$

$$f \text{ is smooth} \Rightarrow \begin{cases} f_{xy} = f_{yx} \\ f_{yz} = f_{zy} \\ f_{zx} = f_{xz} \end{cases}$$

Therefore,

$$\operatorname{curl}(\nabla f) = (0, 0, 0).$$

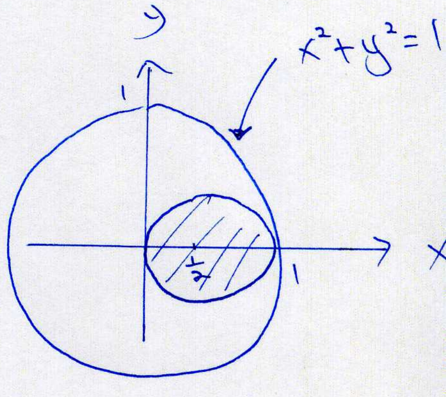
Problem 6 (20 points) Find the volume of the solid that the cylinder $r = \cos\theta$ cuts out of the unit sphere $x^2 + y^2 + z^2 = 1$.



Note that

$$r = \cos\theta \iff r^2 = r\cos\theta$$
$$\iff x^2 + y^2 = x$$
$$\iff \left(x - \tfrac{1}{2}\right)^2 + y^2 = \left(\tfrac{1}{2}\right)^2$$

$\therefore$ the base of the cylinder is a disk centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$.



The solid is

$$E = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{c} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le \cos\theta \\ -\sqrt{1-r^2} \le z \le \sqrt{1-r^2} \end{array} \right\}$$

$$\text{vol}(E) = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} \int_0^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} r\sqrt{1-r^2} \, dr \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \int_{\sin^2\theta}^{1} \sqrt{u} \, du \, d\theta \qquad (u = 1 - r^2)$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} 1 - \sin^3(\theta) \, d\theta$$

$$= \frac{4}{3}\left(\frac{\pi}{2}\right) - \frac{4}{3}\int_0^1 1 - v^2 \, dv \qquad (v = \cos\theta)$$

$$= \frac{2}{3}\pi - \frac{8}{9}$$