

More on limit point and sequential compactness.

For those who are interested, here's "cultural enrichment" material answering some of the questions raised in class on 11/21. If you want more details on anything below, please see me for more discussion or references.

With no additional assumptions,

$$\text{compact} \Rightarrow \text{limit point compact, and} \quad (1)$$

$$\text{sequentially compact} \Rightarrow \text{limit point compact.} \quad (2)$$

Theorem 28.1 is (1). Example 1 in §28 serves as a counterexample to the converses of both (1) and (2).

The added hypotheses needed to prove the converse of (2) are that the space be T_1 (points are closed) and "first countable" (see p. 190). Combined with (1), this implies that compact, first countable, T_1 spaces are sequentially compact. (In the proof of Thm. 28.2, you can see first countability appear as balls of radius $1/i$ and $1/n$. All metric spaces are T_1 and first countable.) Compactness alone does not imply sequential compactness; the standard counterexample is I^I , where $I = [0, 1]$. This product space can also be described as the set of all functions $I \rightarrow I$ with the topology of pointwise convergence.

To go from sequentially compact to compact, we have to strengthen both hypotheses used to reverse the implication in (2) – or else assume, as in Theorem 28.2, that the space is metric. Specifically, if a sequentially compact space either is Hausdorff and second countable or is metric, then it is compact. (Not all metric spaces are second countable, so these are independent sets of hypotheses.) As these hypotheses are stronger than those needed to reverse (2), it follows that a limit point compact space that is either Hausdorff and second countable or metric will also be compact. Sequential compactness alone does not imply compactness; the standard counterexample is the space Munkres denotes S_Ω (p. 66) and most people call "the set of countable ordinals" (with the order topology).

Notice that every result beyond just (1) and (2) requires first countability. Spaces that are Hausdorff but not first countable tend to be fairly complicated (to understand, if not to define, e.g. I^I). This is probably why Munkres used a "grossly non-Hausdorff" counterexample (Example 1, §28) for the converse in Theorem 28.1.

Assignment 10, due Friday, December 5. You should be able to tackle these problems soon enough to turn the assignment in on Wednesday, if the usual due date fits your schedule better, but there will be no late penalty for turning it in up to 4 PM on Friday. If you put it in Josh's box, then send an email noting the day and time you turned it in; 20% penalty if turned in between 4 PM on Friday and 10 AM on Monday, 50% for 10-5 on Monday, and not accepted after that.

I. Reading: §28, and optional reading below in Notes on Surfaces.

II. R problems:

A. Prove that the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ is a connected, compact surface.

B. Prove that the torus T is homeomorphic to the quotient space of Example 5, pp. 139-140.

III. HI problems: p. 181, §28, #2, 3ab, & 6, and the following:

C. Let X be the closed disk $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Define an equivalence relation on X that identifies points on the boundary of the disk with the same y -coordinate; that is, the non-singleton equivalence classes are the pairs of points $(\pm\sqrt{1-y^2}, y)$. Prove X^* is homeomorphic to \mathbb{S}^2 .

Notes on Classification of Compact Surfaces

These notes are very abbreviated, just stating a few definitions, examples, and results. You may find it helpful to read one or more of the following.

- In the book by Patty on reserve, pp. 199-203.
- In Mendelson, *Introduction to Topology*, §5.7, pp. 186-198.
- In Munkres, pp. 446-450 of §74, and §76, pp. 457-462. Ignore references to things we haven't studied (e.g., fundamental group), and skip over details if they are more confusing than helpful. If you're going strong at p. 462, try the rest of the chapter.

The goal for the rest of the quarter is to state and outline a proof of the classification of compact surfaces. First of all, we need to define the terms "classification" and "surface." For a topologist, classification means up to homeomorphism type. That is, a classification of compact surfaces is a list of topological spaces such that every compact surface is homeomorphic to exactly one space on the list.

A surface intuitively is a space that locally looks like the plane, in the same sense that a curve looks like the line. Formally, we define a space to be **locally Euclidean of dimension two** if every point in the space has a neighborhood that is homeomorphic to the open disk

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}. \quad (3)$$

(The space D may be replaced by \mathbb{R}^2 or any open subset of \mathbb{R}^2 without changing the content of the definition; can you prove this?)

We also require surfaces to be Hausdorff, to exclude for instance the plane with multiple origins. Requiring the Hausdorff condition means there are "enough" open sets to exclude certain pathological spaces. At the other extreme, we don't want to allow surfaces to have "too many" open sets. For us, the compactness condition serves this purpose. So we will define a **compact surface** to be a compact Hausdorff space that is locally Euclidean of dimension two.

An aside, for those who are interested: For a surface that isn't necessarily compact, the usual definition is a Hausdorff space that is locally Euclidean of dimension two and is second countable (see p. 190). For locally Euclidean Hausdorff spaces, a space is second countable iff it is metrizable and has a countable number of components. All subspaces of Euclidean space are second countable. Quotient spaces of second countable spaces need not be second countable, but if they are locally euclidean, then they will be second countable. (See me for references and counterexamples if interested.)

The most basic examples of surfaces are \mathbb{R}^2 itself, and every plane in \mathbb{R}^3 . Another general class of examples, including the planes, are graphs of continuous functions $f : U \rightarrow \mathbb{R}$, where U is open in \mathbb{R}^2 . The graph of f is the subspace $\{(x, y, f(x, y)) \in \mathbb{R}^3 \mid (x, y) \in U\}$. To prove it is a surface, show that $F : U \rightarrow \mathbb{R}^3 : (x, y) \rightarrow (x, y, f(x, y))$ is an imbedding. (Of course we can permute the coordinates, and for instance consider the set of points $(f(y, z), y, z)$ as a graph.) This idea can be extended to spaces that are only “locally” graphs by the following lemma.

Lemma. Let S be a subspace of \mathbb{R}^3 that is “locally a graph” at each point. That is, for every $p \in S$, there is a neighborhood V of p in the subspace topology on S so that $V \cap S$ is the graph of a continuous function of two of the three coordinates. Then S is a surface.

Proof. As a subspace of \mathbb{R}^3 , the set S is Hausdorff. By the given conditions, for any $p = (x_0, y_0, z_0) \in S$ we have a neighborhood of p in S of the form $V = \{(x, y, f(x, y)) \mid (x, y) \in U\}$ for some neighborhood U of (x_0, y_0) in the xy -plane (or an analogous neighborhood with the roles of x , y , and z permuted). Because U is open, there is an ϵ -ball B centered at (x_0, y_0) contained in U . By combining an appropriate translation and scaling, B is homeomorphic to the open unit disk D (see (3) above). Let π_{xy} be the projection from \mathbb{R}^3 to the xy -plane, and let $W = \pi_{xy}^{-1}(B) \cap V$. Note that W is also a neighborhood of p in S . Define $F : B \rightarrow W$ by $F(x, y) = (x, y, f(x, y))$. Then F is a continuous bijective function, and it has a continuous inverse, specifically, the restriction of π_{xy} to W . Thus the neighborhood W of p is homeomorphic to B , and thus to D , and we have shown that S is locally euclidean of dimension 2. ■

Note that part of the proof shows that an arbitrary open set U in the plane is itself locally euclidean. By this same argument, we can justify replacing “homeomorphic to D ” in the definition of locally euclidean by “homeomorphic to some open subset of \mathbb{R}^2 .”

As an immediate corollary of the lemma, the sphere \mathbb{S}^2 is a surface. (Consider the six open hemispheres where one of the variables is positive or is negative.) It is also compact. (The easiest way to justify compactness probably is to note it is a level set of the continuous function distance from the origin, which implies that it is both closed and bounded.) Thus we have our first example of a compact surface.

A non-example: The closed upper hemisphere $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}$. This subspace fails to be locally euclidean at points where $z = 0$. It is a *surface with boundary*, but we will not include surfaces with boundary in our classification. Another surface with boundary we have encountered is the Moebius strip, the quotient space of $[0, 1] \times [0, 1]$ induced by identifying $(0, y)$ with $(1, 1 - y)$. You can make this a surface without boundary by removing the “edges”, that is start with $[0, 1] \times (0, 1)$ and make the same identifications.

What *compact* surfaces do we know? Besides the sphere \mathbb{S}^2 , the only other compact surface we have encountered in this course appeared as another quotient of $[0, 1] \times [0, 1]$, Example 5 in §22, pp. 139-140. From that definition, it is possible but tedious to prove the space is locally euclidean. It is easier if we view this space as $\mathbb{S}^1 \times \mathbb{S}^1$, the product of two copies of the circle \mathbb{S}^1 . Mathematicians call this surface the **torus** (plural **tori**). You can picture it as the glazed surface of a doughnut (or if you prefer fewer sweets, the crust of a bagel). Discussion in class: Outline of problems A and B in Assignment 10 (above). (More cultural enrichment: the *n-torus* is the product of n copies of \mathbb{S}^1 , and is locally euclidean of dimension n .)

By identifying the edges of the square $[0, 1] \times [0, 1]$ in other ways, we can obtain other compact surfaces, even the \mathbb{S}^2 ! Discussion in class: The identification which produces \mathbb{S}^2 , and relating this to problem C in Assignment 10 (above).

Shorthand notation for quotients of polygons. The identification space for the torus, given in Example 5 in §22, pp. 139-140, can be represented pictorially by labelling the sides to be identified with a letter and an arrow indicating orientation; see Fig. (i) below. There is a standard way to represent the figure by a sequence of letters. Pick a starting vertex and a direction to travel around the boundary of the figure: we'll start at the lower left vertex and travel counterclockwise. List the label of each edge, and add an "exponent" of -1 if the arrow on the edge points clockwise. Thus Fig. (i) can be abbreviated as $aba^{-1}b^{-1}$.



A space homeomorphic to the sphere results from the quotient indicated in Fig. (ii). Proof outlined in class.

So far, we have been able to use the uniqueness of quotient spaces (Cor. 22.3) to prove the quotients of $[0, 1] \times [0, 1]$ are homeomorphic to spaces we can easily prove are surfaces (the torus and S^2). It is tedious but elementary to prove that the quotient of any $2n$ -gon obtained by pairwise identification of all the edges is a compact surface. (See Thm. 76.1 in Massey, or J. M. Lee, *Introduction to Topological Manifolds*, QA613.2 .L44 2000.) We will assume this result for looking at other quotients of polygons. (Why must we identify the edges pairwise? For example, consider the quotient space indicated by $aaa^{-1}b$; why is it not a surface?)

Figures (iii) and (iv) gives surfaces known as the Klein bottle and the (real) projective plane, respectively. These surfaces cannot be realized as subspaces of \mathbb{R}^3 . Class discussion: orientability; what these surfaces "look like" – as well as we can picture them; other models for the projective plane. Brain teaser: Consider all other possible quotients of $[0, 1] \times [0, 1]$ obtained by identifying sides pairwise. Which ones, if any, give spaces homeomorphic to surfaces we have already considered? Which, if any, give new surfaces?



Connected sums. It is possible to "add" surfaces together as a "connected sum" of surfaces. Here is the process. Cut an open disk – that is, a piece homeomorphic to the open disk – out of each surface. (This process is often described as "putting a hole in the surface." If the surface is a torus, note that this "hole" in the surface is completely different from the "hole in the

doughnut” already in the torus.) Now glue the edges left by removing the disks together, to produce a single surface. In class: Pictures of making the connected sum of two tori; the sphere as the “identity” element for connected sums; the connected sum of two projective planes is a Klein bottle; polygonal representations of these connected sums. The connected sum of g tori is called the *g -hole torus* or *torus* or *handlebody of genus g* . This surface can be imbedded in \mathbb{R}^3 , and is orientable. If we make a connected sum of g projective planes, we obtain a non-orientable surface, sometimes called a *cross-cap of genus g* . (The term *genus* apparently developed just to label the number of summands in the connected sum.)

We can now state the classification theorem.

The classification of compact surfaces. Every connected component of a compact surface is itself a connected, compact surface. Every connected, compact surface is homeomorphic to exactly one of the following:

- The sphere \mathbb{S}^2 .
- The g -hole torus, that is the connected sum of g tori, for some positive integer g .
- The connected sum of g projective planes, for some positive integer g .

(In particular, a g -hole torus is homeomorphic to a \tilde{g} -hole torus if and only if $g = \tilde{g}$, and similarly for connected sums of projective planes.)

There are two major tasks in the proof of this theorem, and we will merely outline them. First, one shows how to construct a homeomorphism for an arbitrary compact surface S with one of the surfaces in the list. The second, harder task is to show that the surfaces in the list are distinct, that is, no two are homeomorphic.

For the first task, the first step is to show that any surface can be “triangulated:” divided into regions homeomorphic to a triangular region in the plane so that the regions overlap in a single mutual edge (or not at all). This result may seem obvious, but subtle in the same way as the Jordan Curve Theorem (that says any simple closed curve divides the plane into exactly two connected open sets). The rest of the process is elementary, but tedious. One argues that the triangles with edges identified may be “pasted together” via edge identifications to a single $2n$ -gon with pairwise identification of all the edges. Finally, one must show that the quotient of the $2n$ -gon is homeomorphic to one of the surfaces in the list, by a process similar to the one used in class to show the connected sum of two projective planes is a Klein bottle.

The second task requires a more advanced version of connectivity called the “fundamental group” of a surface. I will discuss this briefly in class as time permits.