Assignment 2, due Wednesday, April 15.

Reading: pp. 259-263 (start of chapter 8). Also start on the first section of Chapter 7. Don't be scared off by Definition 7.1.6, discussion in class should make it more digestible. You may skip $\S \S 7.1 .3$ and 7.1.6. We'll next do $\S \S 7.2$ and 8.1.3 together.

R Problem: Make a list of formulas for a surfaces of revolution. Some are given on pp. 186-187. Be sure to check the errata (or, do the derivation yourself and correct the formulas as needed). Add the formulas for $K$ and $H$. Also write down the simplified formulas for the case when the generating curve $(f(u), h(u))$ is unit speed and the ones for $h(u)=u$. In particular, show that $K=-f^{\prime \prime} / f$ for the unit speed case, and $K=-f^{\prime \prime} /\left[f\left(1+f^{\prime 2}\right)^{2}\right]$ when $h(u)=u$.

## HI Problems:

Problem S6.3. In addition to adding the hypothesis of compactness, because of multiple questions this problem has been reworded, hopefully more clearly. Also, you may use the Propositions on the other side of this page.

Suppose $S$ is an oriented compact regular surface in $\mathbb{R}^{3}$ and it is on or inside in the sphere of radius $R$ about the point $a \in \mathbb{R}^{3}$; that is, for every point $p \in S$, we have $\|p-a\| \leq R$. Prove that at some point on $S$, the Gaussian curvature $K$ of $S$ is greater than or equal to $1 / R^{2}$.

Remarks: (i) As a corollary, every oriented compact surface in $\mathbb{R}^{3}$ has a point where the Gaussian curvature is positive. (ii) A compact surface is $\mathbb{R}^{3}$ is always orientable, but the proof requires topology beyond $441 / 2 / 3$. Just as the Jordan Curve Theorem says that each simple closed curve separates its complement in the plane into an "inside" and an "outside", a compact surface separates its complement in $\mathbb{R}^{3}$, and we may orient it by the "outward" pointing normal.

Problem 6.6.12, p. 207, modified. Prove that a connected, minimal surface of revolution is contained in a catenoid or one other class of surfaces (which you are to determine).

Hints: First assume that $h^{\prime}(u)$ is never zero, so (justify this!) you may use the $h(u)=u$ version of the formulas. You may assume the following ODE fact: The general solution of the equation $w w^{\prime \prime}=1+\left(w^{\prime}\right)^{2}$ with $w(t)>0$ is $w(t)=(a) \cosh (t / a+b)$. Finally, consider the possibilities that $h^{\prime}(u)$ is always zero or sometimes zero.

Problem S6.5. Let $S$ be the torus given by the parametrization in Exercise 6.5.1, p. 194. The parallels are the $u$-coordinate curves (i.e., $v$ constant). Which parallels are asymptotic curves? Which ones are geodesics? Justify your answers.

Hint: This problem should not involve a lot of computation.

Problem S6.6. Prove that a connected surface of revolution with Gaussian curvature $K=0$ is contained in a cone or a cylinder (a right circular cylinder, not a generalized cylinder).

Two propositions overleaf.

Proposition. Suppose we have a point $a \in \mathbb{R}^{3}$, and a regular curve $\gamma: I \rightarrow \mathbb{R}^{3}$, and the maximum of $\|\gamma(t)-a\|$ occurs at $t=t_{0}$. Then $\gamma\left(t_{0}\right)-a$ and $\gamma^{\prime}\left(t_{0}\right)$ are perpendicular, and $\kappa\left(t_{0}\right) \geq 1 /\left\|\gamma\left(t_{0}\right)-a\right\|$.

The proof is the same as the solution for problem 3 on the Math 442 midterm, with appropriate substitutions of $\gamma(t)-a$ for $\vec{x}(t)$, the curve in that problem.

Proposition. Suppose $S$ is a regular surface and $A$ is a plane perpendicular to $S$ at $p \in S$. Then for some neighborhood $V$ of $p$, the intersection $A \cap S \cap V$ may be parametrized as a regular curve.

This was briefly discussed in class, when we used this result repeatedly in our discussion of normal curvature. For the record, here's an outline of proof.

Sketch of proof. For simplicity, choose coordinates on $\mathbb{R}^{3}$ so that $T_{p} S$ is parallel to the $x y$ plane. Then near $p$ the surface is a graph $z=f(x, y)$, and some equation $a x+b y=c$ will define the vertical plane $A$. Let $(x(t), y(t))$ be a regular parametrization of the line $a x+b y=c$ in the $x y$-plane. Then $(x(t), y(t), f(x(t), y(t)))$ is the desired parametrization of $A \cap S \cap V$.

