Assignment 3, due Wednesday, April 22.

Reading: Add just one page, p. 267, to what was overly-optimistically assigned last week: Ch. 8, pp. 259-263 & p. 267, and start Chapter 7.
Don't be scared off by Definition 7.1.6, discussion in class should make it more digestible.
You may skip §§7.1.3 and 7.1.6.

R Problem: Review the derivation of Equation (8.11) at the bottom of p. 267. Why is the matrix of coefficients antisymmetric?

HI Problems:

Problem S6.7. *Constant positive curvature surfaces of revolution.* In class we considered a surface $X(u,v) = (f(v)\cos u, f(v)\sin u, h(v))$ with constant Gaussian curvature $K = 1/c^2$ and unit speed generating curve $(f(v), h(v))$. We showed that $f(v) = a\cos(v/c)$ for a constant $a > 0$ and

$$h(v) = \int_0^v \sqrt{1 - \frac{a^2}{c^2}\sin^2(t/c)}\, dt$$

(up to vertical shift by adding a constant to h and reparametrizing the generating curve by adding a constant to v). We also analyzed the case $a > c$.

This problem examines the other two possibilities. We restrict attention to $v \in I = [-c\pi/2, c\pi/2)]$, and to the interior of that interval when we consider the surface, so that $f(v) > 0$ (a necessary condition for X to be regular).

(a) Show that if $a = c$, then h is defined for all $v \in I$. Observe that the generating curve for X regular is therefore an open semicircle centered on the axis, so the surface is all of sphere except two points.

(b) Suppose $0 < a < c$. Show that in this case h is also defined for all $v \in I$, so that the length of the generating curve is still $c\pi$. Show that $|h(v)|$ has a maximum of $|h(\pm c\pi/2)| > a$.* Find the slope of the generating curve where it intersects the z-axis at $v = \pm c\pi/2$. (Your formula for the slope should go to zero as $a \to c$ and to infinity as $a \to 0$.) Find the principal curvatures at the points furthest from the z-axis (where the surface intersects the xy-plane).

*Because a is the maximum value of f, showing $|h(\pm c\pi/2)| > a$ means the surface is longer in the z-direction than it is wide in the x-direction. (In fact you will probably show that $|h(\pm c\pi/2)| > c > a$.) This together with the nonzero slope at the z axis gives the surface its (American) "football" shape.

p. 268, Problem 8.1.1.

p. 268, Problem 8.1.2. Orient the cylinder by the outward normal. Add part (c): Find the points on the ellipse where the geodesic curvature κ_g is zero. How could you have predicted that the geodesic curvature would vanish at these points without calculating a formula for κ_g? (Hint: what is the angle between the surface and the plane of the ellipse at those points?)

Problem S8.1. Let S be a regular, oriented surface and γ a regular curve on S.

(a) Prove that

(i) γ is a geodesic $\Leftrightarrow$ $\kappa_g = 0$ and $|\gamma'(t)|$ is constant;

(ii) γ is an asymptotic curve $\Leftrightarrow$ $\kappa_n = 0$; and

(iii) γ is a principal curve $\Leftrightarrow$ $\tau_g = 0$.

Remark. Most of this has been discussed in class. The point of this part of the problem is to add one new observation and bring three similar results together.

(b) Problem 8.2.1, p. 282. Assume γ is nontrivial (to exclude trivial geodesics, that is, curves that have locus = one point). Also, the $\pm$ is incorrect, it's always the same sign; figure out which one it is.

(c) Now let S be the cylinder $x^2 + y^2 = a^2$ oriented by the outward normal, and let $\gamma(t) = (a\cos t, a\sin t, bt)$ for positive a and b. This is the curve in Example 3.2.5 on p. 73, so you may use the calculations on that page.

(i) Find the Darboux frame $\{N, T, U\}$. Show $\kappa_g = 0$ and compute κ_n and τ_g.

(ii) Give formulas for the Frenet apparatus (frame, κ, and τ) in terms of the Darboux frame, κ_g, κ_n, and τ_g.

(iii) How do your results in (i) and (ii) change if you reverse the orientation of S? (You do not need to give an explanation for this, just state the changes.)