Assignment 5, due Wednesday, May 13.

Reading: §§8.2 & 8.3.

HI Problems:

Problem S6.8. Suppose $X(x^1, x^2)$ is a regular patch such that

- The x^1 curves are principal with $\kappa_1 = 0$ everywhere, and
- The x^2 curves are principal with κ_2 never zero.

Prove that the x^1 curves are straight lines in $\mathbb{R}^3$.

Hints. Use the frame $\{N, X_1, X_2\}$. What do the given conditions tell you about N and $L_{ij} = N \cdot X_{ij}$? The desired conclusion is equivalent to X_{11} being a multiple of X_1.

Remarks. (i) If a regular surface has no umbilic points, it can be shown that for every point, one can find a parametrization of a neighborhood with all coordinate lines principal. Thus if a surface is all parabolic - that is, the surface has $K = 0$ everywhere, but one principal curvature nonzero at every point - then the result of this problem says it can only bend along a straight line.

(ii) This problem is numbered S6.8, rather than S8.something, because it only uses earlier ideas. As we didn't get to the Clairaut formula on Friday, I put in this problem and postponed the Clairaut problems until next week.

p. 254, #7.3.8.

Problem S8.3. As a contrast to problem 7.3.8, here is an example of two surfaces with the same Gaussian curvature, but different second fundamental forms, and there is an isometry between them.

Let S_1 be the surface of problem S6.7(b) (from assignment 3) with the parametrization X restricted to $U = \{(u, v) : \pi < u < \pi, -\pi/2 < v < \pi/2\}$. That is, we delete one meridian so X is a regular parametrization. Let S_2 be the sphere of radius c centered at the origin. Find a map $Y : U \to S_2$ that gives a parametrization of part of S_2 so that $F = Y \circ X^{-1}$ is an isometry from S_1 to $Y(U)$ (which is a proper subset of S_2). *Hint:* You may choose F so that it maps each meridian onto a meridian and each parallel to a parallel (but not onto a parallel, or even a parallel less one point).

Remark: Because the two surfaces have different second fundamental forms, there cannot be an isometry of $\mathbb{R}^3$ that maps one surface to the other.

One more on the next page.

Problem S8.4. We continue our exploration of the geodesic torsion τ_g. Suppose $\alpha(t)$ is a unit speed curve in a regular surface.

(a) Assume the curvature κ of α as a space curve is never zero. Prove that

$$\tau = -\tau_g + \frac{\kappa_g \kappa_n' - \kappa_g' \kappa_n}{\kappa^2}.$$

Hints: First express P and B of the Frenet frame in terms of the Darboux frame and the three curvatures. Remember that $\kappa^2 = \kappa_g^2 + \kappa_n^2$.

Remark: Notice that the formula reduces to $\tau = -\tau_g$ if α is a geodesic or an asymptotic curve. Those are the two cases we considered in Problem 8.2.1 and on the midterm.)

(b) Now suppose that κ is identically zero, so α is a straight line. Find an equation relating τ_g and and the Gaussian curvature K.

Hint: Using the Darboux frame as the basis for the tangent space, find the entries in the matrix L_{ij} for the second fundamental form. You should not have to find all of them in order to get the result!