Suggestions for Problem 8.3.2, p. 297: Prove Proposition 8.3.12. The book's hint works, or you can use the suggestion from class to use the Gauss Lemma and the technique used in class to prove it.

Problem 8.3.14, p. 297: Prove Proposition 8.3.14. This proof is a bit tricky because we are asked to do a Taylor expansion for G at $r = 0$, but $G(r, \theta)$ is defined only for r in some interval $[0, \epsilon)$ and the geodesic polar coordinates are not regular at $r = 0$. To completely justify the steps of the proof, some work must be done in Riemannian normal coordinates.

As we will be computing derivatives with respect to r, we may fix the value of θ, and without loss of generality by replacing θ by $\theta - \theta_0$, we may assume $\theta = 0$. Let

$$Y(r, \theta) = \exp_p(r \cos\theta\, w_1 + r \sin\theta\, w_2)$$

be the parametrization giving the geodesic polar coordinates, and let

$$X(u, v) = \exp_p(u\, w_1 + v\, w_2)$$

with the same orthonormal vectors $\{w_1, w_2\}$ give us Riemannian normal coordinates.

In class on Monday 5/18, we will show that $Y_\theta(r, 0) = rX_v(r, 0)$ on $[0, \epsilon)$, and that $f(r) = r||X_v(r, 0)||$ is a C^∞ function of r on some interval including 0, so is equal there to its third order Taylor polynomial with remainder. For $r \in [0, \epsilon)$, we have $\sqrt{G(r, 0)} = f(r)$. Therefore to prove the Proposition, it will suffice to show that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, and $f'''(0) = -K$. That is what you are required to do for the homework.

To find the third derivative, it's helpful to use both coordinate systems. The book's hint says to use Equation (8.32). (That's in §8.3.1, which we skipped. But from the Gauss Lemma, proved in class, one get immediately that formula (7.53) for K reduces to Equation (8.32) in geodesic polar coordinates.) Using that hint, you can find a formula for $f''(r)$ for $r > 0$ (where the polar coordinates really are coordinates!). You must then say why the formula is also valid at $r = 0$, so that you can use it to compute the (one-sided) third derivative.