Supplemental notes will include corrections, additions, and modifications to the text. They will not repeat the corrections on the authors' errata list - see link on the web for this list.

Notation: We will use the abbreviation *p-curve* for *parametrized curve* (definition 1.1.18, p. 5). This is to help us avoid the temptation to say just "curve" when we mean "p-curve." The word "curve" by itself usually means the locus (image) of a p-curve.

Problem clarifications:
Problem 1.1.3: Vector methods will give a simpler proof than the hint given in the book.
Problem 1.1.4: "distance" means "shortest (orthogonal) distance," as in Problem 1.1.3.
Problem 1.2.6: You may assume $\vec{x}(t)$ is regular. At different times, I said to assume $\vec{x}(t)$ is differentiable or that it is regular. If it is only differentiable, then at singular points, the conclusion is still true because the zero vector is interpreted to be perpendicular to all other vectors. Homework grading will allow for either assumption.
Problem 1.3.4: The term "smooth" has not been defined. Here, you may assume that f is three times differentiable.

Smoothness: The term *smooth* has different meanings in different texts. Sometimes it is used as a synonym for differentiable. Sometimes it means C^∞; in practice, this of course means, as many derivatives as you need to take are defined. Our text uses this "practical" definition, so that when a hypothesis says "smooth," the result can later be used in cases where we only assume there are as many derivatives as needed in our calculations. So for instance, in Problem 1.3.4, if we interpret "smooth" to mean C^∞, we would not be able to apply the result later for a function that is only of class C^3, even though it holds in that case.

Proposition 1.2.6.5. (That means, insert between Definition 1.2.6 and Example 1.2.7.)
Reparametrization by arclength. If $\vec{x}(t)$ is a regular p-curve, then there is a reparametrization of $\vec{x}$ by arclength. If $\vec{x}$ is of class C^k, then the arclength reparametrization is also of class C^k.

Proof. We define the arclength $s = f(t)$ as an antiderivative of the speed, $||\vec{x}'(t)||$. As the speed is strictly positive, $f(t)$ is a strictly increasing, so it has an inverse, $t = h(s)$. By the inverse function theorem, because f has nonzero derivative everywhere, its inverse h is also differentiable. The composite function $\vec{y}(s) = \vec{x}(h(s))$ is a reparametrization by arclength: at $s = f(t)$,

$$\vec{y}'(s) = \vec{x}'(h(s))h'(s) = \vec{x}'(t)\frac{1}{f'(t)} = \frac{\vec{x}'(t)}{||\vec{x}'(t)||}.$$

If $\vec{x}$ is of class C^k, then the first derivative of $s = f(t)$ (that is, $||\vec{x}'(t)||$) is of class C^{k-1}, so $f(t)$ is of class C^k. In this case the inverse function theorem also tells us that the inverse function h is of class C^k. The arclength parametrization $\vec{y}(s)$ is therefore the composition of two functions of class C^k and so is also of class C^k. ■

Remark. Strictly speaking, a p-curve parametrized by arclength should have zero in its domain, because arclength is measured from a point on the curve. Note that even though length as a physical quantity is non-negative, it is conventional to extend the definition of arclength $s(t)$ in equation (1.5), p. 14, to values of t that are less than a, which produces negative values of $s(t)$. Having taken this step, the term "parametrized by arclength" is further extended to apply to any unit speed parametrization, even if zero is not in the domain. (In this case $s(t)$ is actually the sum of arclength measured from some point on the curve and a constant.)

Section 1.4. We will use the material on p. 28 as motivation only, and use a simplified version of Corollary 1.4.3 as the definition of *contact of order n*. The motivational material indicates how one could define this concept for curves that are not even differentiable. We will use the concept only for curves that are at least differentiable, and usually C^k for $k \geq 3$, so might as well have a definition in terms of the derivatives of the curves.

In equation (1.14), the notation AD_A means the (Euclidean) distance between A and D_A, and AP^k similarly means the kth power of the distance between A and P. Here n and k are non-negative integers. Note that for $k = 0$, equation (1.14) reduces to $\lim_{A\to P} AD_A = 0$, which holds because C_1 and C_2 intersect at P.

Now let's consider a basic example: C_1 is the locus of (t, t^j) and C_2 is the locus of $(t, 0)$. The point of intersection, P, is the origin. Let $A = (t, t^j)$. Then $D_A = (t, 0)$ and the condition (1.14) becomes

$$\lim_{t\to 0} \frac{|t^j|}{\sqrt{t^2 + t^{2j}}^k} = \lim_{t\to 0} \frac{|t|^j}{|t|^k(1 + t^{2j-2})^{k/2}} = \begin{cases} 1 & k = j \\ 0 & k \leq j - 1 \end{cases}$$

Therefore by the definition in the book, C_2 has contact of order $j - 1$ with C_1. Observe also that the first $j - 1$ derivatives of the two p-curves agree at the origin, but the jth derivative does not.

Motivated by this example, we make the following definition. Let $\vec{\alpha}$ and $\vec{\beta}$ be two regular curves whose loci intersect at P. Assume without loss of generality that $\vec{\alpha}(s)$ and $\vec{\beta}(u)$ are both unit speed with $\vec{\alpha}(s_0) = \vec{\beta}(u_0) = P$. Furthermore, by replacing u by $-u$ if necessary in the formula for $\vec{\beta}$, we may assume that $\vec{\alpha}'(s_0) \cdot \vec{\beta}'(u_0) \geq 0$.

Definition 1.4.3.1 (replacing Definition 1.4.2): Using the notation given above, assume that $\vec{\alpha}(s)$ and $\vec{\beta}(u)$ are C^{n+1}. We say that $\vec{\alpha}$ and $\vec{\beta}$ have *contact of order* n (or, *contact order* n) at P if

$$\vec{\alpha}^{(k)}(s_0) = \vec{\beta}^{(k)}(s_0)$$

for all integers k with $0 \leq k \leq n$, but

$$\vec{\alpha}^{(n+1)}(s_0) \neq \vec{\beta}^{(n+1)}(s_0).$$

It can be shown that this is equivalent to the book's definition. One advantage to the definition 1.4.3.1 above is that it is clearly symmetric with respect to the two curves. (With the definition in the book, we would have either have to prove the symmetry, which is not at all obvious, or be very careful to say C_2 has contact of order n with C_1 when (1.14) is satisfied, but not claim that C_1 has contact of order n with C_2 unless we prove that separately.)

Remark. Notice that in definitions, we say "if" when we really mean "if and only if." It's defininately peculiar that mathematicians insist that "if" and "if and only if" do *NOT* mean the same thing in any other context, but assume the former means the latter in definitions.

Now continue with the text, starting with the paragraph before Definition 1.4.4. The middle sentence of that paragraph needs the following correction: it should end with "and its tangent line have contact order at least 1." (And from the errata on the web - have you made all the corrections in your book yet? - there's an addition to Definition 1.4.4.)

Proposition 1.4.5: Add the assumption that the curve is of class C^3. The statement of the proposition otherwise remains the same, but the proof simplifies tremendously using our definition of contact order.

Proof. To apply our definition, we must reparametrize both the original curve $\vec{x}(t)$ and the proposed osculating circle by arclength. Let $\vec{y}(s)$ be the reparametrization of the original curve $\vec{x}(t)$ by arclength from the point $\vec{x}(t_0)$, so $\vec{y}(0) = \vec{x}(t_0)$. For the proposed osculating circle, again measuring from $\vec{x}(t_0)$, we get

$$\vec{\delta}(s) = \vec{x}(t_0) + \frac{\vec{U}(t_0)}{\kappa_g(t_0)} + \frac{1}{\kappa_g(t_0)}\left(\sin(\kappa_g(t_0)s)\vec{T}(t_0) - \cos(\kappa_g(t_0)s)\vec{U}(t_0)\right).$$

(The unit tangent and normal and the curvature are those from the original curve $\vec{x}(t)$.) Differentiating and evaluating at $s = 0$, we find that the 0th, 1st, and 2nd derivatives of $\vec{y}$ and $\vec{\delta}$ agree, so these curves have contact order at least two.

It remains to check uniqueness. If we have any other unit speed circle that passes through $\vec{x}(t_0)$ with tangent parallel to $\vec{T}(t_0)$, its center must be on the line $\vec{x}(t_0) + \lambda\vec{U}(t_0)$, where $|\lambda|$ is the radius of the circle, and $1/\lambda$ is the curvature of the circle. (Think through why this is correct if $\lambda < 0$.) If the circle has contact order at least two with our original curve, this means $\lambda = 1/\kappa_g(t_0)$, so the osculating circle is uniquely determined. ∎