Some revision has been made to comments on the proof of the Isomperimetric Inequality from the first version posted for these notes.

## Remarks on $\S 1.5$

In $\S 1.5$, the terms "rigid motion," "positive isometry," and "composition of rotations and translations" are used as if they are synonymous. The full set of rigid motions for the plane is the set of transformations that are compositions of reflections as well as rotations and translations. Reflections are excluded from consideration here because they reverse orientation, and thus do not preserve the cross product and also reverse the sign of the curvature.

By definition, an isomety is a mapping from one metric space to another that preserves the distance function:

$$
F:\left(X_{1}, d_{1}\right) \rightarrow\left(X_{2}, d_{2}\right) \text { such that } d_{2}(F(x), F(y))=d_{1}(x, y) \text { for all } x, h \in X_{1} .
$$

(Here $\left(X_{i}, d_{i}\right)$ is a set $X_{i}$ with a metric (distance function) $d_{i}$.) The require that the metric be preserved implies that an isometry is continuous. If the metric comes from a scalar product on Euclidean space, a mapping that preserves the scalar product must be an isometry. Conversely, Supplementary Problem S2.2 (at the end below, or see Assignment 3) shows that an isometry of Euclidean space must be the sum of a translation and a mapping that preserves the scalar product.

Here is a slight rewording of Theorem 1.5.2, the Fundamental Theorem of Plane Curves. Let $\kappa_{g}: I \rightarrow \mathbb{R}$ be a piecewise continuous function on an interval $I$. Then
(a) there exists a unit speed parametrized curve $\vec{x}: I \rightarrow R^{2}$ with curvature function $\kappa_{g}$; and (b) This curve is unique up to a positive isometry.

## Chapter 2

Note that the Definition 2.1.1, p.39, implicitly requires that a regular curve be infinitely differentiable, at least at the endpoints of the domain (because all the derviatives must be defined there to make sense of the stated conditions). But the idea is that the image of the endpoints should be "just as nice" as the other points on the curve. So for consistency we will assume that regular means the curve is $C^{\infty}$.

In Green's Theorem, 2.1.3, and its Corollary 2.1.4, the curve does not need to be $C^{\infty}$. The following will suffice: Assume the curve is a positively oriented, simple closed p-curve that is piecewise regular and $C^{1}$. This means that the domain can be divided into a finite number of closed subintervals with the parametrization regular and $C^{1}$ (using one-sided derivatives at the endpoints) on each subinterval.

The Isomperimetric Inequality, Theorem 2.3.1, should have the same conditions on the curve as in Green's theorem.

Some comments on the proof. One should prove that $\bar{y}(s)$ is $C^{1}$ (though no book I've checked does this). The parametrization $\gamma(s)$ for the circle may not be a simple closed curve, so Green's Theorem as stated does not apply. However, if the curve fails to be simple, it go "back and forth" over a section of the circle, and the area integral $-\int \bar{y} d x=$
$-\int x^{\prime}(s) \bar{y}(s) d s$ will compute the area between the curve and the axis positively when moving counterclockwise and negatively when moving clockwise. So after cancellation, we will get the correct area enclosed by the curve. (Contrast this to the case of arclength, where the distance along the curve is counted as positive whichever direction the curve is parametrized.) To go from the first to the second line of (2.6), add the nonnegative quantity $\left(x x^{\prime}+\bar{y} y^{\prime}\right)^{2}$ under the radical. (I previously said the geometric arithmetic mean inequality was need here, but it is not.) To prove the geometric mean is no greater than the arithmetic mean, that is, $\sqrt{a b} \leq(a+b) / 2$ for nonnegative $a$ and $b$, start with $(\sqrt{a}-\sqrt{b})^{2} \geq 0$.

In Definition 2.4.1, p. 53, the last sentence should say, "If $S$ [not $C$ ] is not convex, ...". The original sentence with $C$ and the reference to Figure 2.5 should be on the next page at the end of Definition 2.2.2.

Propositions 2.4.3 and 2.4.4. Because the definition of convex requires a curve to be simple, it does not change the content of the first proposition to include "simple" in the initial list of conditions on the curve. We can always reparametrize a regular closed curve to be unit speed and counterclockwise around its interior. So let us rephrase the results of these two propositions as follows.

Let $\vec{x}: I \rightarrow R^{2}$ be a simple, regular, closed, unit speed parametrized curve that goes counterclockwise around its interior, and let $C$ be its locus. Then the following three conditions are equivalent:
(i) The curvature of $\vec{x}$ is nonnegative at every point. (If it were parametrized clockwise instead, the curvature would instead be nonpositive.)
(ii) The curve is convex, that is, the set $S$ that is the union of $C$ and its interior is a convex set.
(iii) At every point of $C$, all of $C$ is on one side of the tangent line to $C$ at that point. (Here the "side" includes its boundary, that is, the tangent line.)

Discussion. The condition that the curve be simple is necessary for our text's definition. Challenge: Contruct (picture, not formulas) a nontrivial example of a closed curve that is not simple, but the union of the curve $C$ with points in the bounded connected components of $\mathbb{R}^{2}-C$ is a convex set. ("Nontrivial" excludes a failure to be simple because the curve "goes around more than once," e.g., ( $\cos t, \sin t$ ) on $[0,4 \pi]$.) Most books take (iii) as the definition of convex. One can then prove that the curve is simple, so it satisfies the definition of convex in our text.

Proof. We will prove the contrapositives of the equivalences; that is, we will show that the following are equivalent.
(i) The curvature of $\vec{x}$ is negative at some point.
(ii) The curve fails to be convex.
(iii) At some point of $C$, the tangent line has points of $C$ on one both sides.

For any point $p$ on $C$, by applying an isometry we may assume that $p$ is the origin, the unit tangent at $p$ is $(1,0)$, and therefore from our counterclockwise assumption, the interior $C$ includes points on the positive $y$-axis. To show ( $\mathrm{i}^{\prime}$ ) implies (ii'), let $p$ be a point where the curvature is negative, and apply an isometry as just described. Near $p$, we may reparametrize
$C$ as a graph $(t, f(t)$. Using the formula for curvature you computed for Problem 1.3.5, we see that negative curvature at $p$ implies that the curve has a strict local maximum for the $y$-coordinate at $p$. Thus near near $p$, the curve lies in the 3rd and 4th quadrants. A chord between two points near $p$ on $C$, one in each of these quadrants, will cross the negative $y$-axis arbitrarily close to the origin, which will be in the exterior of $C$. Therefore $C$ is not convex.

For (ii') implies (iii'), if the curve fails to be convex, then there will be a pair of points $p$ and $q$ on the curve such that the line segment $L$ between $p$ and $q$ includes no points in the interior of $C$. As above, we may assume that $p$ is the origin and the unit tangent at $p$ is $(1,0)$. Therefore from our counterclockwise assumption, the interior $C$ includes points on the positive $y$-axis. As the interior of $C$ is bounded by $C$, there must be some points of $C$ above the $x$-axis.

We will now show that $q$ must be on or below the $x$-axis. Reparametrize the curve near $p$ by $(t, f(t)$. Using the Taylor Theorem with Remainder for the p-curve, we know that there are positive numbers $\varepsilon$ and $M$ such that for $t<\varepsilon$, we have $|f(t)|<M t^{2}$. Thus the interior of $C$ includes an open set $V$ with lower boundary $M t^{2}$ for $t<\varepsilon$. If $q$ is above the $x$-axis, then the line segment $L$ will intesect $V$, contradicting the definition of $L$.

If $q$ is below the $x$-axis, which is the tangent line at $p$, then we have points on $C$ both above and below that tangent line and have shown (iii'). If $q$ is on the $x$-axis, WLOG on the positive $x$-axis, then the definition of $L$ means the points $(x, 0)$ between $p$ and $q$ must be in the exterior of $C$, so the curve goes into the first quadrant from $p$. Therefore there is a point on $C$ near $p$ whose tangent line has positive slope. Then $p$ is on one side of that tangent line, and $q$ is on the other. So we have that (ii') implies (iii').

For (iii') implies (i'), I do not see a way to fill in the gap in the text's method of proof. Instead, we will use Proposition 2.2.8, which says that the rotation index of a simple, closed regular curve $\pm 1$. We will prove this proposition later, with no use of Propositions 2.4.3 or 2.4.4, so no circular reasoning. If you would like to see the proof now, see Theorem 2, p. 396, in the do Carmo book on reserve (same title as our book).

This is a proof by contradiction. We assume (iii'), that there are points of $C$ on both sides of the tangent line at $p$, and (i), that the curvature is nonnegative. As before, we may assume that tangent line is the $x$-axis, so we have points on the curve with positive $y$-coordinate and with negative $y$-coordinate. Because the p-curve is continuous, by the Extreme Value Theorem there points $q$ and $r$ where $y$ has its maximum and minumum values of $y$, respectively. The tangent lines at those points are also horizontal. Using the positive orientation, we find that the unit tangent vectors at $p$ and $r$ are both $(1,0)$, while at $q$ the unit tangent must be $(-1,0)$. If $\vec{T}=(\cos \theta, \sin \theta)$, then at $p$ and $r$, we have $\theta$ is an multiple of $2 \pi$, while at $q$ it is an odd integral multiple of $\pi$. Recall (p. 36) that $\kappa_{g}=d \theta / d s$. For contradiction, we assumed the curvature is nonnegative, so the angle $\theta$ is nondecreasing. No matter the order in which the curve passes through the three points, this means that $\theta$ changes by at least $2 \pi$ between $p$ and $r$, and then must change some more to complete the closure of the curve. This contradicts Proposition 2.2.8, showing that (i) is false; therefore (iii') implies (i').

See next page for correction to 2.4.2 and supplementary problems.

## Chapter 2 Problem correction:

Problem 2.4.2 (a): $\kappa$ should be $\kappa_{g}$.

## Supplementary problems.

Problem S2.1. Compute the area of an an ellipse with semi-axes of lengths $a$ and $b$. (Hint: You should be able to do this by both of the following methods.
(i) Make a change of variables so the ellipse becomes a circle, and apply the change of variables formula for integrals to the integral that gives the area. We will need the change of variables formula in Chapter 6.
(ii) Use Corollary 2.1.4.)

Problem S2.2. Let $\left(\mathbb{R}^{n}, d\right)$ represent $\mathbb{R}^{n}$ as a metric space with the distance function $d$ between points defined from the scalar product in the standard way:

$$
d(\vec{A}, \vec{B})=\sqrt{(\vec{A}-\vec{B}) \cdot(\vec{A}-\vec{B})}
$$

Suppose that $F:\left(\mathbb{R}^{n}, d\right) \rightarrow\left(\mathbb{R}^{n}, d\right)$ is an isometry, i.e., $d(F(\vec{A}), F(\vec{B}))=d(\vec{A}, \vec{B})$. Prove that $F$ preserves the scalar product:

$$
F(\vec{A}) \cdot F(\vec{B})=\vec{A} \cdot \vec{B} \text { for all } \vec{A}, \vec{B} \in \mathbb{R}^{n}
$$

As we remarked in class, the converse is easy to prove. Thus isometry for Euclidean spaces may be defined either by requiring preservation of the distance function or preservation of the scalar product.

Problem S2.3. Use the Isoperimetric Inequality to prove that the arclength $L$ of an ellipse with semi-axes of lengths $a$ and $b$ satisfies inequality $L \geq 2 \pi \sqrt{a b}$.

Problem S2.4. Construct a nontrivial example of a closed curve that is not simple, but the union of the curve $C$ with points in the bounded connected components of $\mathbb{R}^{2}-C$ is a convex set. You need not give explicit formulas, but must explain how the formulas could be constructed. "Nontrivial" excludes a failure to be simple because the curve travels around the same locus twice or more, e.g., $(\cos t, \sin t)$ on $[0,4 \pi]$.

