(Correction to S3.3(a).)

Many of the clarifications in this supplement make assumptions explicit that are arguably in the text, but only by implication. To make sure we don't overlook them, it seems better to make them explicit. If you notice additional implicit assumptions that aren't discussed here, let me know. Also remember that corrections listed in the official errata list for the book (link under "**Text**" on the course homepage) are not repeated in supplemental notes.

Clarifications and Corrections.

Example 3.2.5 (Helicies), p. 73, should specify $a > 0$. Otherwise we should change the formula for the curvature κ to have $|a|$ instead of just a in the numerator.

Proposition 3.2.7, p. 75, part 2, assumes (implicitly, but let's make it explicit) that the torsion exists everywhere on the curve. This is equivalent to assuming the curature κ never vanishes.

In **Exercise 3.2.4(b))**, p. 76, assume that the original curve is C^3 as well as regular, and that the reparametrization is also regular and C^3 for both parts of the problem. Change part (b) to say, "Prove that the torsion function τ is unchanged under the reparametrization." That is, show that τ, and not just its absolute value, is unchanged.

Exercise 3.2.6, p. 75, change the last line to say, "Prove that for a planar curve, wherever its torsion is defined, that torsion is zero." (Otherwise, the exercise could be interpreted to ask you to prove that the torsion exists and is zero everywhere on the curve, which is not true. You should be able to give an example of a planar curve whose torsion does not exist at any point, and an example where the torsion exists at some points and not at others.)

Exercise 3.2.10(c), p. 77: Delete the words "or is planar" at the end of the exercise (because this possibility is excluded by the assuption of nonzero torsion at the start of the problem).

Figure 3.5(b), p. 79, is incorrect. The arrow on the curve is supposed to indicate the direction of parametrization, for otherwise this curve is the same as in (a). So both $\vec{T}$ and $\vec{B}$ are incorrect, and this curve has *positive* torsion. Be sure you understand why all three of these corrections must be made, and what a correct example of negative torsion would look like.

Proposition 3.3.2, p. 79, assumes (implicitly) that the curvature κ is nonzero at $t = t_0$. (If the curvature vanishes, the osculating plane is not defined. In this case, there may or may not be a plane that has contact order two or greater, and if it exists, it may not be unique.)

Some additional terminology and exposition.

We shall refer to the curvature functions, the torsion function, and the Frenet frame of a curve as the *Frenet apparatus* of the curve. Thus if you are asked to compute the Frenet apparatus of a curve, you must find the three vectors of the Frenet frame and the two functions. The term *Frenet equations* will refer to the equations for the derivatives of the Frenet frame, that is, (3.4), p. 72, which reduces to (3.11), p.75, if the speed is identically equal to one.

Here is a sumary of the key results in §3.2 (excepting the discussion of helices). Assume a curve $\vec{x}$ is regular and C^2, so that its unit tangent vector $\vec{T}$ is defined and differentiable. Then we can define the curvature κ for the curve. We also have

$$\vec{T}' \neq \vec{0} \Leftrightarrow \kappa \neq 0 \Leftrightarrow \vec{x}' \times \vec{x}'' \neq \vec{0}.$$

If we further know that one, and thus all, of the conditions in the preceeding display hold, and that the curve is C^3, then all of the Frenet apparatus of $\vec{x}$ is defined. Usually the most convenient formulas for computing this apparatus are (3.1) (definition of $\vec{T}$), (3.7) (for κ), (3.9) (for τ),

$$\vec{B} = \frac{\vec{x}' \times \vec{x}''}{||\vec{x}' \times \vec{x}''||}, \text{ and } \vec{P} = \vec{B} \times \vec{T}.$$

Supplementary problems.

Problem S3.1. Let $\vec{x}(t) = \cosh(t), \sinh(t), t)$.

(a) Compute the Frenet apparatus for $\vec{x}$.

(b) Show that $\vec{x}$ is a helix.

(c) Find the unit vector $\vec{u}$ for the helix $\vec{x}$.

(d) Generalize your calculation to find a formula for the unit vector $\vec{u}$ for any helix $\vec{x}$ in terms of the Frenet apparatus of $\vec{x}$.

Problem S3.2. Let

$$f(t) = \begin{cases} e^{-1/t} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0. \end{cases}$$

Prove that $f'(0) = 0$, $f''(0) = 0$ and $f'''(0) = 0$. Note that you cannot do this by computing these derivatives for $t \neq 0$ and taking the limit of the derivative; you must compute each derivative as the limit of a difference quotient. Can you see why the derivatives of f to all orders at the origin will exist and be equal to zero? If not, continue computing derivatives until the pattern is clear to you.

We know from the formulas of first year calculus that f has derivatives of all orders for $t \neq 0$, so with the results of this problem, we know f is C^∞. It is, however, not real analytic. If you have taken complex analysis, think about what happens if you try to extend this function to a complex differentiable function. What goes wrong?

Problem S3.3. Using the function f from Problem S3.2, define a parametrized curve $\vec{x}(t) = (t, f(t), f(-t))$.

(a) Find where the curvature of $\vec{x}$ is zero.

(b) Show that the torsion of $\vec{x}$ is zero wherever it is defined.

(c) Why might someone think this curve is a counterexample to part 2 of Proposition 3.2.7? Why is it not a counterexample?

Problem S3.4. Let $\vec{\alpha}(t)$ be a regular curve in $\mathbb{R}^3$ such that the curvature κ never vanishes. Suppose that the acceleration $\vec{\alpha}''(t)$ is constant; that is, $\vec{\alpha}''(t) = \vec{W}$, where $\vec{W}$ is a constant vector. Express $\vec{\alpha}''(t)$ in terms of the Frenet apparatus of $\vec{\alpha}$ and differentiate it to prove the following three results.

(a) The torsion τ of $\vec{\alpha}$ is identically zero.

(b) The speed $s'(t)$ of $\vec{\alpha}$ has positive second derivative $s'''(t)$.

(c) The derivative of speed, $s''(t)$, and the derivative of curvature, $\kappa'(t)$, have opposite signs. That is, prove that if one of these derivatives is positive, then the other one is negative.

Remarks, not part of the problem. This problem has the following physical interpretation. The curve is the path of an object acted on by a uniform force field proportional to $\vec{W}$ at every point in space. Part (a) says the path of the object lies in a plane. Part (c) says that the curvature of the path is inversely related to the speed: the curvature decreases of the speed is increasing, and the curvature increases if the speed is decreasing. Which situation holds in a particular case depends on the relation between the field and the initial velocity.