Here's a summary of our work in class and the rest of the solution.

The problem. Find a steady state solution for the temperature in a cylinder of radius 2 and height 10 with temperature zero on the curved surface and the bottom and temperature $f(r,\theta)$ on the top. That is, solve

$$r^2 u_r r + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} + u_{zz} = 0 \tag{1}$$

$$u(r,\theta,10) = f(r,\theta), \qquad u(r,\theta,10) = 0 = u(2,\theta,z). \tag{2}$$

We separated these equations, and also noted some implicit boundary conditions on the separated equations, to get

Problem for Θ:

$$\Theta'' - B\Theta = 0, \qquad \Theta(-\pi) = \Theta(\pi), \qquad \Theta'(-\pi) = \Theta'(\pi);$$

Problem for R:

$$r^2 R'' + rR' + (Ar^2 + B)R = 0, \qquad R(0+) \text{ is bounded}, \qquad R(2) = 0.$$

Other separated equations:

$$Z'' + AZ = 0, \qquad Z(0) = 0.$$

Soving first Sturm-Liouville Problem. The Problem for Θ already is a Sturm-Liouville problem, with eigenvalue $= -B$, so we solve that first. The solution is a set of eigenvalues and eigenfunctions :

$$-B = n^2, n = 0, 1, 2, ..., \Theta_0(\theta) = 1, \Theta_n(\theta) = a_n \cos(\theta) + b_n \sin(theta)$$

Soving second Sturm-Liouville Problem. Now the Problem for R becomes a Sturm-Liouville problem, with eigenvalue $= A$:

$$r^2 R'' + rR' + (Ar^2 - n^2)R = 0, \qquad R(0+) \text{ is bounded}, \qquad R(2) = 0.$$

Our work at the start of the chapter lead to the conclusion that the only solutions for the differential equation that are bounded at zero are the Bessel Functions (of the first kind), $J_n(\mu r)$ with $A = \mu^2$. Then the other boundary condition says $J_n(2\mu) = 0$, so we have

$$A = \left(\frac{\lambda_{n,k}}{2}\right)^2, \qquad R_{n,k}(r) = J_n(\lambda_{n,k} r/2),$$

where $\lambda_{n,k}$ is the k-th positive zero of J_n ($k \in \mathbb{N}$

Some comments, not required in the solution, but because of questions people asked. In earlier problems, one boundary condition (often at zero) would tell us we only get sine functions, or only get cosine functions. The $R(0+)$ bounded condition is working in a similar way: it tells us we only get the J_n functions, not the Y_n functions. If we let $A < 0$, the solution is actually a "modified Bessel function" that is similar to the hyperbolic sine and cosine, so can't solve a homogeneous boundary condition at $b > 0$. You can read a little about these "modified Bessel functions" in §5.6 if you are interested.

Solving the other separated equations. Using $A = (\lambda_{n,k}/2)^2$ from solving the R equations, we get

$$Z_{n,k}(z) = \sinh(\lambda_{n,k} z/2).$$

Putting the solution together from the solutions of the separated equations:

$$u(r,\theta,t) = \sum_{k=1}^{\infty} a_{0,k}[J_0(\lambda_{0,k} r/2)\sinh(\lambda_{0,k} z/2) + \sum_{n=1}^{\infty} J_n(\lambda_{n,k} r/2)\sinh(\lambda_{n,k} z/2)(a_{n,k}\cos n\theta + b_{n,k}\sin n\theta)].$$

Finding coefficients, variations on the boundary conditions. Setting $z = 10$, we get

$$f(r,\theta) = \sum_{k=1}^{\infty} a_{0,k}[J_0(\lambda_{0,k} r/2)\sinh(5\lambda_{0,k}) + \sum_{n=1}^{\infty} J_n(\lambda_{n,k} r/2)\sinh(5\lambda_{n,k})(a_{n,k}\cos n\theta + b_{n,k}\sin n\theta)].$$

We find the coefficients, as usual, by taking inner products, that is, integrals, of f and the products of eigenfunctions, but don't forget the sinh factors! Here are the answers, in terms of the formulas for $c_{n,k}$ and $d_{n,k}$ given at the bottom of p. 151:

$$\begin{aligned} a_{n,k} &= c_{n,k}/\sinh(5\lambda_{n,k}) && \text{for } n = 0,1,2,\ldots \\ b_{n,k} &= d_{n,k}/\sinh(5\lambda_{n,k}) && \text{for } n = 1,2,\ldots \end{aligned}$$

Note the difference between the factor in front of the integral for $c_{0,k}$ and the one for $c_{n,k}$ for positive n. You should know where that factor of 2 came from.

If the given function f is independent of θ, then it will be orthogonal to all the functions $J_n(\lambda_{n,k} r/2)(a_{n,k}\cos n\theta + b_{n,k}\sin n\theta)$ for $n > 0$, and we only get the first, single summation term in the solutions. Suppose instead $f(r,\theta) = y = r\sin(\theta)$. Again most of the coefficients will be zero; which ones won't be?

Finally, what if the boundary condition on the curved surface is changed to $u_r(2,\theta,z) = 0$? This changes $\lambda_{n,k}$ values to the first zero of the *derivative* of J_n (represented in Theorem 5.3, p., 147, with a "tilde" over the λ). We also get one extra eigenvalue for R when $n = 0$: we get $A = 0$ with the constant function as the eigenfunction. Exercise: Find the corresponding function $Z(z)$ and the term that must be added to the summation formula for u in this case.