On Wednesday, April 25, the discussion of the last part of the proof of Theorem 3.5 was a little rushed, and was supposed to be a model for you for one of the homework problems. So here is the argument (with more detail than would have been possible in class, even with more time).

We had discussed the proof of the first sentence of the theorem, about bases for $L^2(-\pi,\pi)$. To prove the set $\{\sin nx\}_{n=1}^{\infty}$ is an orthogonal basis for $L^2(0,\pi)$, we consider an arbitrary function $f \in L^2(0,\pi)$ and extend f to an odd function $\tilde{f} \in L^2(-\pi,\pi)$. (If necessary, change the value at $x = 0$ to 0; we've already agreed that functions that differ only at a finite number of points are "the same" in L^2.) By the first result in the proof, we know that $\tilde{f}$ is equal in norm to its Fourier series (in cosines and sines). But we constructed $\tilde{f}$ to be an odd function, so the coefficients of the constant term and all the cosine terms will be zero. Furthermore, the sine function coefficients for $\tilde{f}$ are

$$\begin{aligned} b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\sin(nx)dx &= \frac{\langle \tilde{f}, \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle} \qquad (1)\\ = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx &= \frac{\langle f, \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle}. \qquad (2)\end{aligned}$$

In line (1), the inner products are in $L^2(-\pi,\pi)$, while in (2), the inner products are in $L^2(0,\pi)$. (The formula for the coefficient as the ratio of two inner products is in equation (3.23), which is the generalization to infinite dimensions of the formula you prove in section 3.1, #2.)

Now we will show that the set $\{\sin nx\}_{n=1}^{\infty}$ is complete in $L^2(0,\pi)$ (that is, that condition (b) in Theorem 3.4 holds) by showing that condition (a) in Theorem 3.4 holds for these sine functions on $[0,\pi]$. Let $f \in L^2(0,\pi)$ be a function such that for all $n \in \mathbb{N}$,

$$\langle f, \sin nx\rangle = \int_0^{\pi} f(x)\sin(nx)dx = 0.$$

We have to show that f is equal (in norm) to the zero function; that is that

$$||f||^2 = \int_0^{\pi} (f(x))^2 dx = 0.$$

So we extend f to an odd function $\tilde{f}$. By the results of the preceding paragraph, all the Fourier series coefficients for $\tilde{f}$ vanish: the a_n's by symmetry, and the b_n's because they are the same as those for f, which are zero by assumption. But we know from the first result in the theorem that the set of sines and cosines is an orthogonal basis for $L^2(-\pi,\pi)$. By Theorem 3.4 (a), this implies

$$||\tilde{f}||^2 = \int_{-\pi}^{\pi} |\tilde{f}(x)|^2 dx = 0.$$

By symmetry, $||f||^2 = \frac{1}{2}||\tilde{f}||^2$, so $||f||^2 = 0$, as we wanted to show. Thus (a) in Theorem 3.4 holds for the set $\{\sin nx\}_{n=1}^{\infty}$ in $L^2(0,\pi)$. By the theorem, this means (b) in the theorem also holds; but this says $\{\sin nx\}_{n=1}^{\infty}$ is a basis for $L^2(0,\pi)$.