Here is a discussion of finding the general solution for Problem N, and some remarks about how this relates to finding the coefficients for the boundary conditions given. Both of these may be helpful for some final exam problems. After reading this, if you should recheck (or completely recompute, if needed) the coefficients for the given boundary conditions. Various versions of the final answer will be posted on Catalyst.

Problem N. §4.4, p. 120, #7 with the following modifications: set $\beta = \pi$ and $r_0 = e^{-2}$, and let $g(r) = 5\sin(4\pi \log r)$ and $h(r) = 3$. Write your final answer as simply and explicitly as possible.

Discussion. The mathematical BVP we are asked to solve is

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0,$$
$$u(r_0, \theta) = 0 = u(1, \theta),$$
$$u(r, 0) = 5\sin(4\pi \log r), \quad \text{and} \quad u(r, \pi) = 3,$$

for $r \in [r_0, 1]$ for positive r_0 and $\theta \in [0, \pi]$.

The regular Sturm-Liouville problem that comes from separating variables for this problem is

$$rR'' + R' + \frac{\lambda R}{r} = 0, \qquad R(e^{-2}) = 0 = R(1).$$

The other separated equation we obtain is $\Theta'' - \lambda\Theta = 0$.

The Sturm-Liouville problem is almost identical to the one in problem in §3.5, #10. The only difference is that the interval has the form $[b, 1]$ (instead of $[1, b]$ that appeared in 3.5.10). There are no solutions with $\lambda \le 0$, and for $\lambda > 0$, the boundary conditions lead to solutions

$$R_n(r) = \sin(n\pi \ln(r)/2), \qquad \lambda_n = (n\pi/2)^2, n \in \mathbb{N}.$$

You could express these functions as a sum of functions of the form r^{ic}, where c is some real number. If the inhomogeneous conditions were given as exponentials or complex-valued functions, there might be an advantage to giving the eigenfunctions in a form using exponentials instead of trig functions. In this problem, the conditions are given in terms of real-valued functions, including $\sin(4\pi \log r)$. This clearly indicates that for ease in computing coefficients, we should use $\sin(n\pi \ln(r)/2)$ instead of an exponential form.

For each value of n, the equation for Θ_n now gives us a pair of linearly independent solutions that can be expressed in terms of exponentials or in terms of hyperbolic trig functions. If you consult the answer in the back of the book, you're encouraged to use exponentials. Or you might choose the usual pair of hyperbolic functions, $\cosh(n\pi\theta/2)$ and $\sinh(n\pi\theta/2)$. If you split the problem into two problems, in each problem setting one of the two inhomogeneous boundary conditions at $\theta = 0$ and $\theta = \pi$ to zero, you'll get functions that vanish at those values of θ. That is, you'll get $\sinh(n\pi\theta/2)$ and $\sinh(n\pi(\pi - \theta)/2)$. Here are the three resulting formulas for the general solution:

$$
\begin{aligned}
u(r,\theta) &= \sum_{n=1}^{\infty} \sin(n\pi \ln(r)/2)[a_n e^{n\pi\theta/2} + b_n e^{-n\pi\theta/2}] && (1)\\
&= \sum_{n=1}^{\infty} \sin(n\pi \ln(r)/2)[A_n \cosh(n\pi\theta/2) + B_n \sinh(n\pi\theta/2)] && (2)\\
&= \sum_{n=1}^{\infty} \sin(n\pi \ln(r)/2)[c_n \sinh(n\pi\theta/2) + d_n \sinh(n\pi(\pi-\theta)/2)]. && (3)
\end{aligned}
$$

Or any other general form of the solution for $\Theta_n(\theta)$ can be used.

For (1) (or (2)) each of the two boundary conditions at $\theta = 0$ and $\theta = \pi$ will give you a linear equation for a_n (or A_n) and b_n (or B_n) for each n. Then for each n you solve the pair of linear equations. This is simpler than it sounds because the boundary condition at $\theta = \pi$ is a constant function. For (3), setting θ to zero eliminates the terms with with c_n, and setting it to π eliminates the d_n terms.

If you had something close to (1), (2), or (3), check or recompute your coefficients for that form of the answer. If not, pick one now and compute the coefficients. Answers will be posted on Canvas.