In the first two classes after the midterm, we worked through Exercise **3.5.5**, finding the eigenvalues and respective eigenfunctions for the regular Sturm-Liouville (SL) problem

$$(A)\, f'' + \lambda f = 0, \qquad (B)\, f'(0) = 0, \qquad \text{and} \qquad (C)\, f'(l) = \beta f(l).$$

Here is a brief summary of our work and the final conclusions.

Recall the physical interpretation of the boundary condition at $x = l$: $\beta = 0$ means an insulated boundary, while $\beta < 0$ models heat radiating from the end $x = l$ according to Newton's Law of Cooling, with ambient temperature $= 0$. The case $\beta > 0$ is not physically reasonable, because it would mean that when the end is hotter than than its surroundings, heat flows *into* the object, but we can still solve the SL problem.

For convenience of computation, for each $\lambda$, we consider all possible (real) values of $\beta$. The final answer is organized differently: given a particular value of $\beta$, we want to know a complete set of eigenfunctions and corresponding eigenvalues. We also want to know the norms of the eigenfunctions (for use in computing coefficients for eigenfunction expansions).

We found that when $\lambda = 0$, we get a nontrivial solution only if $\beta = 0$. (It's the constant solution we have found before when both ends are insulated.)

If $\lambda = -\mu^2 < 0$, then (A) and (B) imply an eigenfunction must be of the form $f(x) = \cosh(\mu x)$. Plugging this function into (C) and doing some algebra produces

$$\tanh \mu l = \frac{\beta}{\mu}. \tag{1}$$

(Note we assumed $\mu^2 \neq 0$, so it's OK to divide by $\mu$.) By graphing both sides of this equation as functions of $\mu$, we saw that there is no solution if $\beta \leq 0$, and a pair of solutions $\mu_0, -\mu_0$ if $\beta > 0$. But $\pm\mu_0$ give the same $\lambda$ and also the same $f$, so we have one (linearly independent) eigenfunction $f(x) = \cosh(\mu_0 x)$ with eigenvalue $-\mu_0^2$.

For $\lambda = \nu^2 > 0$, then (A) and (B) imply an eigenfunction must be of the form $f(x) = \cos(\nu x)$. If $\beta = 0$, we have the familiar case $\nu = n\pi/l$ for integers $n$, so the eigenvalues are $\lambda_n = [n\pi/l]^2$ for positive integers $n$. If $\beta \neq 0$, then (C) implies $\nu$ is a nonzero solution to

$$\tan \nu l = -\frac{\beta}{\nu}. \tag{2}$$

Again graphing the two sides of the equation, this time as functions of $\nu$, we saw that there are an infinite sequence of solutions for every (nonzero) value of $\beta$. If $\nu$ is a solution, then $-\nu$ is also a solution; but we get the same $\lambda$ and same eigenfunction for $\pm\nu$, so we restrict attention to positive $\nu$. We also saw that as the $n \to \infty$, the solutions will be approximately the values where $\tan(\nu l) = 0$; that is, $\nu \approx n\pi/l$. More precisely, for large $n$, if $\beta < 0$, then $\nu_{n+1} \gtrapprox n\pi/l$; and if $\beta > 0$, then $\nu_n \lessapprox n\pi/l$. Thus for large $n$ the eigenvalues and eigenfunctions are almost the same as for the insulated case ($\beta = 0$).

Now we summarize and compute the norms. For $\beta = 0$, the eigenvalues are $\lambda_n = [n\pi/l]^2$ for $n = 0, 1, 2, ...$, and the corresponding eigenfunctions are $f_n(x) = \cos(n\pi x/l)$. We know from previous work that $||f_n||^2 = l/2$.

If $\beta < 0$, every eigenvalue has the form $\nu_n^2$ with $\nu_n$ the *nth* positive solution to (2). There are an infinite number of these, with values going to infinity. The corresponding eigenfunctions are $f_n(x) = \cos(\nu_n x)$, with

$$\begin{aligned} ||f_n||^2 = \int_0^l \cos^2(\nu_n x)dx &= \frac{l}{2} + \frac{\sin(2\nu_n l)}{4\nu_n} = \frac{l\nu_n + \sin(\nu_n l)\cos(\nu_n l)}{2\nu_n} \\ &= \frac{l\nu_n \sin(\nu_n l) + \sin^2(\nu_n l)\cos(\nu_n l)}{2\nu_n \sin(\nu_n l)} \\ &= \frac{l\beta - \sin^2(\nu_n l)}{2\beta} \end{aligned} \tag{3}$$

(using the fact that $\nu_n \sin(\nu_n l) = -\beta\cos(\nu_n l)$ to get the last line). Any of these expressions for the square of the norm is correct, and the book uses one more,

$$||f_n||^2 = \frac{l|\beta| + \sin^2(\nu_n l)}{2|\beta|}.$$

The last two expressions show that for large $n$, $\sin^2(\nu_n l) \approx 0$, so $||f_n||^2 \approx l/2$.

If $\beta > 0$, we still get the eigenvalues $\lambda_n = \nu_n^2$ with $\nu_n$ the *nth* positive solution to (2), corresponding eigenfunctions are $f_n(x) = \cos(\nu_n x)$, and norm-squared as in (4). In addition, we get one additional eigenvalue, $\lambda = -\mu_0^2$ where $\mu_0$ is the single positive solution to (1), with eigenfunction $\cosh(\mu_0 x)$. It only remains to compute the square of its norm:

$$\int_0^l \cosh^2(\mu_0 x)dx = \frac{2l\mu_0 + \sinh(2\mu_0 l)}{4\mu_0} = \frac{l\beta + \sinh^2(\mu_0 l)}{2\beta},$$

(using hypertrig identities and the fact that $\beta/\mu_0 = \tanh \mu_0 l$).

In all cases, by Theorem 3.10, the complete set of eigenfunctions is an orthogonal basis for $L^2(0, l)$.

*Additional remark on the physics.* The eigenvalues for the physically reasonable case $\beta < 0$ are all positive. Recall the factors $\exp(-k\lambda_n t)$ in the solution u(x,t): all positive eigenvalues means the solution will decay to zero. This fits the situation we are modelling: as heat radiates, the solution $u(x, t)$ should tend to to zero.