Problem 0-1, designed to inoculate you (I hope!) from making some common errors about continuity and differentiability in several variables.

Reminder of some definitions. A function f on an open subset of a product space $X \times Y$ is said to be *continuous separately on* X or *continuous separately in* x if for each $y_0 \in Y$, the function $x \mapsto f(x, y_0)$ is continuous on X wherever defined. Separate continuity on Y or in y is defined similarly. The definitions for differentiability and directional derivatives of a function on $\mathbb{R}^n$ are given on p. 642 and p. 647 of ISM.

Problem: Give an example of each of the following. (This of course means prove they are examples. You may find some of the theorems in ISM Appendix C useful in writing the proofs.)

(a) A function $f(x, y)$ on an open set in $\mathbb{R}^2$ that is continuous in each variable separately but is fails to be continuous at one or more points in its domain.

(b) A function $f(x, y)$ on an open set in $\mathbb{R}^2$ that is continuous everywhere and has a directional derivative in every direction at every point of its domain, but fails to be differentiable at one or more points in its domain.

Remarks, not part of the problem.

Note that partial derivatives are special cases of directional derivatives. Thus part (b) shows that differentiability is a strictly stronger condition than the existence of partial derivatives.

Similarly part (a) says that proving separate continuity in every variable is not sufficient to prove that a function is continuous.

You also should know of the existence of a function whose mixed second partials are not equal. An example is provided in most calculus and advanced calculus texts.