Problem 5-19. If G is a smooth manifold with a group structure such that the multiplication map m is smooth, prove that G is a Lie group.

The only feature of a Lie group that G lacks is smoothness of the inversion map $i : G \to G$, $i(g) = g^{-1}$. Here are two and a half proofs that i is smooth.

First proof, not using hint. Define $F : G \times G \to G \times G$ by $F(g, h) = (g, gh)$. By given assumptions, F is smooth, and has a set function inverse, $F^{-1}(g, k) = (g, k^{-1}g)$. We show below that F in fact is a diffeomorphism, so F^{-1} is smooth. Then $i(g) = F^{-1}(e, g)$ is also smooth.

To see that F is a diffeomorphism, we first show it is a submersion. Using the characterization of the tangent space for a product manifold given in Problem 3-2, the tangent space at each point of $G \times G$ itself may be regarded as a product space. Using Prop. 3.17 to compute the action of dF shows that $dF(X_g, 0) = (X_g, dR_h(X_g))$ and $dF(0, Y_h) = (0, dL_g(Y_h))$. Combining, we get $dF(X_g, Y_h) = (X_g, dR_h(X_g) + dL_g(Y_h))$. For this map, we can compute an inverse: For $k = gh$,

$$(dF)^{-1}(X_g, Z_k) = (X_g, L_{g^{-1}} Z_k - L_{g^{-1}} dR_{g^{-1}k}(X_g)$$

(Note that left or right multiplication by a fixed element of G is smooth, even if that element is an inverse!) Thus F is a bijective submersion, so by Theorem 4.8(c), it is a diffeomorphism. ■

Second proof, using hint. Actually, Problem 4-3 doesn't apply directly, because it assumed that G is a Lie group. But if you check the solution of 4-3, you will see that the smoothness of the inverse function is never used. (We need that left or right multiplication by the inverse of a fixed element is smooth, but that doesn't require smoothness of inversion.)

So, by the solution of Problem 4-3, m is a submersion. Therefore by the Submersion Level Set Theorem, Cor. 5.12, the set $S = m^{-1}(e) = \{(g, g^{-1}) : g \in G\}$ is an embedded submanifold of $G \times G$, with dimension equal to that of G.

By Prop. 5.23, the projections π_1 and π_2 from $G \times G$ to the first and second factors, respectively, restrict to smooth maps $\pi_j|_S : S \to G$. We prove below a lemma that $\pi_1|_S$ has constant rank. It is also bijective, so by Theorem 4.8 it is a diffeomorphism. Thus it has a smooth inverse, call it $\varphi : G \to S : g \mapsto (g, g^{-1})$. Composing this with π_2, we conclude that $i = \pi_2 \circ \varphi$ is smooth.

Lemma. The map $\pi_1|_S$ has constant rank.

Proof of the lemma. We show in fact that $d\pi_1|_S$ is surjective at each point of S. Let $\sigma : S \to G \times G$ be the inclusion map. By Exercise 5.22, the image of $d\sigma_s$ is the kernel of dm_s. We want to show that $d(\pi_1 \circ \sigma)_s$ is surjective. By dimension count, it suffices to show that

$$\operatorname{rank} d(\pi_1 \circ \sigma)_s = \dim G = \operatorname{rank} d\sigma_s.$$

By Exercise B.20(d), this follows if we show that

$$\{0\} = \operatorname{Im} d\sigma_s \cap \ker (d\pi_1)_s = \ker dm_s \cap \ker (d\pi_1)_s.$$

Let $X \in \ker(d\pi_1)_s$, where $s = (g, g^{-1}) \in S$. We can represent X by a curve of the form $(g, \gamma(t))$, where $\gamma(0) = g^{-1}$ and $\gamma'(0) = d\pi_2(X)$. Suppose X is in the kernel of dm_s. Then

$$dm(X) = \left.\frac{d}{dt}\right|_{t=0} m(g, \gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} L_g(\gamma(t)) = d(L_g)(\gamma'(t)) = 0.$$

This holds only if $\gamma'(t) = 0$, because L_g is a diffeomorphism. Thus $\ker dm_s \cap \ker(\pi_1)_s = \{0\}$, as required. ■

And the half proof, a second proof of the lemma for the second proof.

Consider the left actions $L_g : G \to G$ and $L_g \times R_{g^{-1}} : G \times G \to G \times G$. Restrict the latter action to S:

$$L_g \times R_{g^{-1}}(k, k^{-1}) = (gk, k^{-1}g^{-1}) = (gk, (gk)^{-1}).$$

Note it preserves S, and also is transitive on S. Furthermore, $\pi_1|_S$ is equivariant with respect to the two actions:

$$L_g \circ \pi_1|_S(k, k^{-1}) = gk = \pi_1|_S \circ L_g \times R_{g^{-1}}(k, k^{-1}).$$

Thus by the Equivariant Rank Theorem, Theorem 4.43, $\pi_1|_S$ has constant rank. ■

(It's easy to show that $d\pi_1|_S$ is surjective at (e, e), using Problem 3-5, so this approach also can be used to show π_1 is a submersion.)