Problem 17-12. (a) For the special case, the solution is $u(t,s) = \varphi(s) + \int_0^t f(r,s)dr$, where $(t,s) \in \mathbb{R} \times S$. By the Fundamental Theorem of Calculus this is a solution. If u_1 and u_2 are both solutions, then $u = u_1 - u_2$ satisfies the initial value problem $u|_S \equiv 0$ and $\partial u/\partial t = 0$. By the Mean Value Theorem, this implies $u \equiv 0$ on $\mathbb{R} \times S$.

For the general case, the idea is to construct a diffeomorphism ψ from a neighborhood of S in $\mathbb{R} \times S$ to a neighborhood of S in M such that $\psi(0,s) = s$ and $d\psi(\partial/\partial t) = N$. Let $\mathcal{D}$ be the domain of the flow of N, and let θ be the restriction of that flow to $U = \mathcal{D} \cap (\mathbb{R} \times S)$. Then θ is a local diffeomorphism at points in the preimage of S because N is transverse to S. Furthermore, we can find a neighborhood V of $\{0\} \times S$ in U for each $s \in S$, there is only one t such that $\theta(t,s) \in S$. (This fact is essentially a rephrasing of Corollary 7.20.) Define $V^{(s)} = \{t | (t,s) \in V\}$. By further restriction if necessary, we may assume that each $V^{(s)}$ is symmetric with respect to the origin.

First we restrict to $V \subset U$ so that for $(t,s) \in V$, we have $\theta(t,s) \in S$ only for $t = 0$. We also want θ to be a local diffeomorphism everywhere on V and $V^{(s)} = \{t | (t,s) \in V\}$ equal to an interval containing the origin. Because S is embedded and θ is a local diffeomorphism, for every $s \in S$ we can find a neighborhood of $(0,s)$ satisfying these conditions, then take the union of these local domains V_s. Restricting further if necessary, we may assume each $V^{(s)}$ is symmetric with respect to the origin.

Now define W to be "V with all the time intervals halved." Formally, let $W = \{(t,s) \in V | (2t,s) \in V\}$, and define $\psi = \theta|_W$. By construction of V, we have that ψ is a local diffeomorphism, so it only remains to show that it is one-to-one. So suppose that $\theta(t_1,s_1) = \theta(t_2,s_2)$ for $(t_1,s_1),(t_2,s_2) \in W$. Without loss of generality, suppose that $|t_2| \le |t_1|$. Let $(-a,a) = V^{(s_1)}$ (where a may be infinity), so $|t_1| < a/2$. Then $|t_1 - t_2| < a$, so $(t_1 - t_2, s_1) \in V$. Applying the flow we have

$$\theta(t_1 - t_2, s_1) = \theta(-t_2, \theta(t_1,s_1)) = \theta(-t_2, \theta(t_2,s_2)) = \theta(0,s_2)) = s_2.$$

By the construction of V, this implies that $t_1 - t_2 = 0$ and so $s_2 = s_1$ and $t_1 = t_2$. Therefore ψ is a diffeomorphism.

By construction, ψ is defined on a neighborhood of $\{0\} \times S$, its image is a neighborhood of S in M and $\psi(0,s) = s$. Because ψ is a restriction of the flow of N, we have $\psi_*(\partial/\partial t) = N$. Let $\tilde{u}$ be the solution to the problem

$$\tilde{u}(0,s) = \varphi(s) \text{ and } \frac{\partial \tilde{u}}{\partial t}(s,t) = f(\psi(t,s))$$

on a neighborhood of $\{0\} \times S$, and define $u = \tilde{u} \circ \psi^{-1}$. The initial condition $u|_S = \varphi$ is immediate. For $m = \psi(s,t)$, we have

$$N_m u = \psi_*(\frac{\partial}{\partial t}|_{(s,t)})u = \frac{\partial}{\partial t}(u \circ \psi)(s,t) = \frac{\partial}{\partial t}(\tilde{u})(s,t) = f(\psi(t,s)) = f(m).$$

Thus u is a solution to the original problem. If there were two distinct solutions u_1 and u_2, then $u_1 \circ \psi$ and $u_2 \circ \psi$ would be distinct solutions to the problem on $\mathbb{R} \times S$, contradicting the special case. Therefore u is the unique solution. ■

(b) The submanifold S is the x-axis. By simple integration the flow of $N = y\partial/\partial x + \partial/\partial y$ is $\theta(t,s) = (s + t^2/2, t)$, which inverts to $\theta^{-1}(x,y) = (x - y^2/2, y)$. The solution to the problem on $\mathbb{R} \times S$ is

$$\tilde{u}(t,s) = \sin(s) + \int_0^t (s + \frac{r^2}{2})dr = \sin(s) + st + \frac{t^3}{6}.$$

Composing this with ψ^{-1} produces the solution for the original problem,

$$u(x,y) = \tilde{u}(x - \frac{y^2}{2}, y) = \sin(x - y^2/2) + (x - \frac{y^2}{2})y + \frac{y^3}{6} = \sin(x - y^2/2) + xy - \frac{y^3}{3},$$

which is easily checked by differentiating.