MATH 309B
FINAL SOLUTIONS
June 9, 2008

1. (25 points) Find the solution to the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} \ , \ \mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} .$$

SOLUTION The first step is to compute the eigenvalues of our matrix.

$$\det \begin{pmatrix} -\lambda & 1 & -1 \\ -1 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{pmatrix} = -\lambda \det \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} - \det \begin{pmatrix} -1 & 1 \\ 0 & -\lambda \end{pmatrix} =$$

$$-\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).$$

Thus, the eigenvalues are $0, \pm\imath$. The next step is compute the eigenvectors associated to each eigenvalue. For $\lambda = 0$, we solve

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{\xi}_0 = \mathbf{0}$$

to get

$$\boldsymbol{\xi}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} .$$

For $\lambda = \imath$, we solve

$$\begin{pmatrix} -\imath & 1 & -1 \\ -1 & -\imath & 1 \\ 0 & 0 & -\imath \end{pmatrix} \boldsymbol{\xi}_\imath = \mathbf{0}$$

by placing it an augmented matrix and reducing.

$$\left(\begin{array}{ccc|c} -\imath & 1 & -1 & 0 \\ -1 & -\imath & 1 & 0 \\ 0 & 0 & -\imath & 0 \end{array}\right) \overset{R_1+\imath R_3, R_2-\imath R_3}{\longrightarrow} \left(\begin{array}{ccc|c} -\imath & 1 & 0 & 0 \\ -1 & -\imath & 0 & 0 \\ 0 & 0 & -\imath & 0 \end{array}\right) \overset{R_1-\imath R_2}{\longrightarrow}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -\imath & 0 & 0 \\ 0 & 0 & -\imath & 0 \end{array}\right)$$

so we can take

$$\boldsymbol{\xi}_\imath = \begin{pmatrix} \imath \\ -1 \\ 0 \end{pmatrix} .$$

We know that an eigenvector, $\boldsymbol{\xi}_{-\imath}$, with eigenvalue $-\imath$ is just the conjugate of $\boldsymbol{\xi}_{\imath}$. So,

$$\boldsymbol{\xi}_{-\imath} = \begin{pmatrix} -\imath \\ -1 \\ 0 \end{pmatrix}.$$

Thus, the general (complex-valued) solution to our system of ODEs is

$$\boldsymbol{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \imath \\ -1 \\ 0 \end{pmatrix} e^{\imath t} + c_3 \begin{pmatrix} -\imath \\ -1 \\ 0 \end{pmatrix} e^{-\imath t}.$$

To get a general real-valued solution, we take the real and imaginary parts of

$$\begin{pmatrix} \imath \\ -1 \\ 0 \end{pmatrix} e^{\imath t} = \begin{pmatrix} -\sin(t) \\ -\cos(t) \\ 0 \end{pmatrix} + \imath \begin{pmatrix} \cos(t) \\ -\sin(t) \\ 0 \end{pmatrix}$$

to get

$$\boldsymbol{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -\sin(t) \\ -\cos(t) \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} \cos(t) \\ -\sin(t) \\ 0 \end{pmatrix}.$$

To solve the initial value problem, we need to choose c_1, c_2, c_3 so that

$$\boldsymbol{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

We can take $c_1 = c_2 = c_3 = 1$ to get our solution

$$\boldsymbol{x}(t) = \begin{pmatrix} 1 - \sin(t) + \cos(t) \\ 1 - \cos(t) - \sin(t) \\ 1 \end{pmatrix}.$$

2. (40 points) Find the general solution to the nonhomogeneous system of ODEs

$$\mathbf{x}' = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-t} \\ e^{t} \end{pmatrix}.$$

SOLUTION We first find the general solution to the corresponding homogeneous system. We compute eigenvalues and eigenvectors.

$$\det \begin{pmatrix} 3-\lambda & 4 \\ -1 & -1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 1.$$

So the only eigenvalue is $\lambda = 1$. We see what the possible eigenvectors are by solving

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}.$$

The only possible solutions are

$$\boldsymbol{\xi}_1 = c \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Thus, we do not have a basis of eigenvectors, but we do have one solution

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$$

The other solution, for our fundamental set, is of the form

$$(\boldsymbol{\eta}_1 t + \boldsymbol{\eta}_0) e^t.$$

Substituting into the differential equations and equating coefficients we need to solve

$$\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} \boldsymbol{\eta}_1 = \boldsymbol{\eta}_1$$
$$\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} \boldsymbol{\eta}_0 = \boldsymbol{\eta}_1 + \boldsymbol{\eta}_0.$$

$\boldsymbol{\eta}_1$ is a eigenvector so we must have

$$\boldsymbol{\eta}_1 = c \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

To solve for $\boldsymbol{\eta}_0$, we put everything into an augmented matrix and row reduce.

$$\begin{pmatrix} 2 & 4 & | & 2c \\ -1 & -2 & | & -c \end{pmatrix} \overset{R_1+2R_2}{\longrightarrow} \begin{pmatrix} 0 & 0 & | & 0 \\ -1 & -2 & | & -c \end{pmatrix}$$

So

$$\boldsymbol{\eta}_0 = \begin{pmatrix} c - 2y \\ y \end{pmatrix}$$

For some c and y. We can take $c = 1$ and $y = 0$ to get

$$\boldsymbol{\eta}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus, the general solution to homogeneous equation is

$$\boldsymbol{x}(t) = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t + c_2 \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \right).$$

We now need a single solution to the inhomogeneous equation. Since there is not a basis of eigenvectors for our coefficient matrix, we cannot use the method of diagonalization to get this solution. We use the two remaining methods to get the solution.

First, we use the method of undetermined coefficients. We split our inhomogeneous term into

$$\begin{pmatrix}1\\0\end{pmatrix} e^{-t} + \begin{pmatrix}0\\1\end{pmatrix} e^{t}$$

and consider

$$\mathbf{x}' = \begin{pmatrix}3 & 4\\-1 & -1\end{pmatrix}\mathbf{x} + \begin{pmatrix}1\\0\end{pmatrix} e^{-t}$$

and

$$\mathbf{x}' = \begin{pmatrix}3 & 4\\-1 & -1\end{pmatrix}\mathbf{x} + \begin{pmatrix}0\\1\end{pmatrix} e^{t}$$

separately, find solutions to each, and superpose them.

For the first differential equation, we guess our solution is of the form $\boldsymbol{\alpha} e^{-t}$. Subsituting this ansatz into the differential equation, we need to solve

$$\begin{pmatrix}3 & 4\\-1 & -1\end{pmatrix}\boldsymbol{\alpha} = -\boldsymbol{\alpha} - \begin{pmatrix}1\\0\end{pmatrix}.$$

We put this into augmented form and get

$$\left(\begin{array}{cc|c}4 & 4 & -1\\-1 & 0 & 0\end{array}\right).$$

Thus,

$$\boldsymbol{\alpha} = \begin{pmatrix}0\\-1/4\end{pmatrix}.$$

For the second differential equation, we guess our solution is of the form

$$\left(\frac{1}{2}\boldsymbol{\beta}_2 t^2 + \boldsymbol{\beta}_1 t + \boldsymbol{\beta}_0\right) e^{t}.$$

Subsituting into the differential gives the following equations for the $\boldsymbol{\beta}$:

$$\begin{aligned}
\begin{pmatrix}3 & 4\\-1 & -1\end{pmatrix}\boldsymbol{\beta}_2 &= \boldsymbol{\beta}_2\\
\begin{pmatrix}3 & 4\\-1 & -1\end{pmatrix}\boldsymbol{\beta}_1 &= \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2\\
\begin{pmatrix}3 & 4\\-1 & -1\end{pmatrix}\boldsymbol{\beta}_0 &= \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 - \begin{pmatrix}0\\1\end{pmatrix}.
\end{aligned}$$

We have solved the first two equations already.

$$\boldsymbol{\beta}_2 = c\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
$$\boldsymbol{\beta}_1 = \begin{pmatrix} c-2y \\ y \end{pmatrix}.$$

To solve for $\boldsymbol{\beta}_0$, we look again at the augmented matrix and reduce.

$$\begin{pmatrix} 2 & 4 & | & c-2y \\ -1 & -2 & | & y-1 \end{pmatrix} \xrightarrow{R_1+2R_2} \begin{pmatrix} 0 & 0 & | & c-2 \\ -1 & -2 & | & y-1 \end{pmatrix}.$$

For this to have a solution, we need $c = 2$. We can take $y = 1$ and $\boldsymbol{\beta}_0 = \mathbf{0}$. Thus, a solution to

$$\mathbf{x}' = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$$

is

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} t^2e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} te^t.$$

The general solution to

$$\mathbf{x}' = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-t} \\ e^t \end{pmatrix}$$

is

$$\boldsymbol{x}(t) = c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t + c_2\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^t + \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} e^{-t} + \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t\right) e^t.$$

Now, we find a solution using the method of variation of parameters. The fundamental matrix is

$$X(t) = \begin{pmatrix} 2e^t & 2te^t + e^t \\ -e^t & -te^t \end{pmatrix}.$$

Its inverse is

$$X^{-1}(t) = e^{-2t}\begin{pmatrix} -te^t & -2te^t - e^t \\ e^t & 2e^t \end{pmatrix} = \begin{pmatrix} -te^{-t} & -2te^{-t} - e^{-t} \\ e^{-t} & 2e^{-t} \end{pmatrix}.$$

The solution we seek is

$$X(t)\int_{t_0}^t X^{-1}(s)\begin{pmatrix} e^{-s} \\ e^s \end{pmatrix} ds.$$

Let us compute the integral first.

$$\int_{t_0}^t \begin{pmatrix} -se^{-s} & -2se^{-s} - e^{-s} \\ e^{-s} & 2e^{-s} \end{pmatrix}\begin{pmatrix} e^{-s} \\ e^s \end{pmatrix} ds = \int_{t_0}^t \begin{pmatrix} -se^{-2s} - 2s - 1 \\ e^{-2s} + 2 \end{pmatrix} ds =$$
$$\begin{pmatrix} -\frac{1}{2}se^{-2s} - \frac{1}{4}e^{-2s} - s^2 - s \\ -\frac{1}{2}e^{-2s} + 2s \end{pmatrix}\Bigg|_{t_0}^t.$$

We set $t_0 = 0$ and get

$$\begin{pmatrix} \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} - t^2 - t + \frac{1}{4} \\ -\frac{1}{2}e^{-2t} + 2t + \frac{1}{2} \end{pmatrix}.$$

Then,

$$X(t)\int_0^t X^{-1}(s)\begin{pmatrix} e^{-s} \\ e^{s} \end{pmatrix} ds = \begin{pmatrix} 2e^t & 2te^t + e^t \\ -e^t & -te^t \end{pmatrix}\begin{pmatrix} \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} - t^2 - t + \frac{1}{4} \\ -\frac{1}{2}e^{-2t} + 2t + \frac{1}{2} \end{pmatrix} =$$

$$\frac{1}{4}\begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} t^2 e^t + \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \frac{1}{4}\begin{pmatrix} 4 \\ -1 \end{pmatrix} e^t.$$

The general solution is then

$$\boldsymbol{x}(t) = c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t + c_2\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^t +$$

$$\begin{pmatrix} 0 \\ -1/4 \end{pmatrix} e^{-t} + \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} t^2 + \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} t + \begin{pmatrix} 1 \\ -1/4 \end{pmatrix}\right) e^t.$$

3. (25 points) Determine if the following statements are true or false. DO NOT JUSTIFY YOUR ANSWERS.

   (a) One can use the method of diagonalization to get a solution to the following inhomogeneous systems of ODEs:

   $$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} t^{-1} \\ 5t \end{pmatrix}.$$

   SOLUTION: FALSE. There is not a basis of eigenvectors.

   (b) The integral

   $$\int_0^{\pi} \sin(x)\sin(nx)dx$$

   is zero for all $n = 0, 1, 2, \ldots$

   SOLUTION: FALSE. The integral is zero for $n \neq 1$, but nonzero for $n = 1$.

   (c) The Neumann problem on any region Ω with $\dfrac{\partial u}{\partial \eta} = 0$ on $\partial\Omega$ has only the trivial solution (i.e. $u = 0$).

   SOLUTION: FALSE. $u = c$ for any constant c is a solution.

   (d) The method of separation of variables applied to the PDE

   $$u_{xx} + u_x u_t + u_{tt} = 0$$

   gives two independent ODEs in t and x.

   SOLUTION: FALSE.

(e) For any constant coefficient, linear, homogeneous system of ODEs, one can write down an explicit fundamental set of solutions.
SOLUTION: TRUE.

4. (20 points) Find the fundamental solutions to the heat equation

$$u_{xx} = u_t$$

with boundary conditions

$$u(t,0) = 0 = u_x(t,\pi)$$

where $0 \leq t < \infty$ and $0 \leq x \leq \pi$. Use them to solve the heat equation with the above boundary conditions and initial data

$$u(0,x) = \sin(\frac{3x}{2}).$$

SOLUTION: We use separation of variables to find the fundamental set of solutions. If we assume that $u(t,x) = T(t)X(x)$ is a solution, we get the two ODEs

$$X'' + \lambda X = 0$$
$$T' + \lambda T = 0.$$

If we incorporate the boundary conditions, we need

$$u(t,0) = T(t)X(0) = 0$$
$$u_x(t,\pi) = T(t)X'(\pi) = 0.$$

So $X(0) = X'(\pi) = 0$. If $\lambda < 0$, set $\lambda = -\mu^2$. Then, the general solution to $X'' + \lambda X = 0$ is

$$X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

and $X(0) = c_1 + c_2$, $X'(\pi) = \mu(c_1 e^{\mu\pi} - c_2 e^{-\mu\pi})$. Our boundary conditions force $c_1 = -c_2$ and then $e^{\mu\pi} + e^{-\mu\pi} = 0$. e^x is positive for any real x so we have no nonzero solutions in this case. If λ is zero, then the general solution is

$$X(x) = c_1 + c_2 x$$

and $X(0) = c_1$, $X'(\pi) = c_2$. Thus, in this case, there are no nonzero solutions. If $\lambda > 0$, set $\lambda = \mu^2, \mu > 0$. Then, the general solution is

$$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

and $X(0) = c_1$ and $X'(\pi) = \mu c_2 \cos(\mu\pi)$. We need $c_1 = 0$ and $\cos(\mu\pi) = 0$. So $\mu = m - 1/2$ for $m = 1, 2, \ldots$ So $\lambda = (m - 1/2)^2$. Solving $T' + (m - 1/2)^2 T = 0$ we get our fundamental set of solutions

$$u_m(t,x) = e^{-(m-1/2)^2 t} \sin((m - 1/2)x).$$

If we take $m = 2$, then we see that $u_2(t,x)$ solves the heat equation with $u_2(t,0) = 0$ and $(u_2)_x(t,\pi) = 0$ and $u_1(0,x) = \sin(\frac{3x}{2})$.

5. (20 points) Solve the wave equation

$$u_{xx} = u_{tt}$$
$$u_x(t,0) = u_x(t,\pi) = 0$$
$$u(0,x) = 0 \ , \ u_t(0,x) = g(x)$$

with $0 \le t < \infty, 0 \le x \le \pi$ and initial data

$$g(x) = \begin{cases} x & 0 \le x < \pi/2 \\ \pi - x & \pi/2 \le x \le \pi. \end{cases}$$

Write your solution as a linear combination of left and right moving waves.

SOLUTION: In this case, we know that the solution is

$$u(t,x) = \frac{1}{2}k_0 t + \sum_{n=1}^{\infty} k_n \sin(nt)\cos(nx)$$

where

$$k_n = \frac{2}{n\pi}\int_0^\pi g(x)\cos(nx)dx$$

for $n > 0$ and

$$k_0 = \frac{2}{\pi}\int_0^\pi g(x)dx.$$

We compute k_0 first.

$$k_0 = \frac{2}{\pi}\left(\int_0^{\pi/2} xdx + \int_{\pi/2}^{\pi}(\pi - x)dx\right) = \frac{2}{\pi}\left(\frac{x^2}{2}\Big|_0^{\pi/2} + (\pi x - \frac{x^2}{2})\Big|_{\pi/2}^{\pi}\right) = \frac{\pi}{2}.$$

Now, we compute k_n for $n > 0$.

$$k_0 = \frac{2}{n\pi}\left(\int_0^{\pi/2} x\cos(nx)dx + \int_{\pi/2}^{\pi}(\pi - x)\cos(nx)dx\right).$$

$$\int x\cos(nx)dx = \frac{1}{n}x\sin(nx) + \frac{1}{n^2}\cos(nx)$$

so

$$\int_0^{\pi/2} x\cos(nx)dx = \frac{\pi}{2n}\sin(\frac{n\pi}{2}) + \frac{1}{n^2}\cos(\frac{n\pi}{2}) - \frac{1}{n^2}.$$

And,

$$-\int_{\pi/2}^{\pi} x\cos(nx)dx = -\frac{1}{n^2}\cos(n\pi) + \frac{\pi}{2n}\sin(\frac{n\pi}{2}) + \frac{1}{n^2}\cos(\frac{n\pi}{2}),$$

$$\int_{\pi/2}^{\pi} \pi\cos(nx)dx = -\frac{\pi}{n}\sin(\frac{n\pi}{2})$$

so

$$k_n = \frac{2}{n^3\pi}(2\cos(\frac{n\pi}{2}) - \cos(n\pi) - 1).$$

Thus,

$$k_n = \begin{cases} 0 & n = 4m \\ 0 & n = 4m+1 \\ \frac{-8}{n^3\pi} & n = 4m+2 \\ 0 & n = 4m+3. \end{cases}$$

$$u(t,x) = \frac{1}{2}\left(\frac{k_0}{2}(x+t) + \sum_{n=1}^{\infty} k_n \sin(n(x+t)) - \left(\frac{k_0}{2}(x-t) + \sum_{n=1}^{\infty} k_n \sin(n(x-t))\right)\right)$$

is the decomposition into a superposition of left and right moving waves.

6. (20 points) Solve the following Dirichlet problem in the unit disc $\{(x,y) \mid x^2+y^2 \leq 1\}$.

$$\Delta u(r,\theta) = 0,$$

$$u(1,\theta) = \begin{cases} 0 & 0 \leq \theta < \pi \\ 1 & \pi \leq \theta < 2\pi. \end{cases}$$

SOLUTION: We know that

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos(n\theta) + k_n \sin(n\theta))$$

where

$$c_n = \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\cos(n\theta)d\theta$$

and

$$k_n = \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\sin(n\theta)d\theta.$$

We first compute c_0.

$$c_0 = \frac{1}{\pi}\int_{\pi}^{2\pi} d\theta = 1.$$

Now we compute c_n.

$$c_n = \frac{1}{\pi}\int_{\pi}^{2\pi} \cos(n\theta)d\theta = \frac{1}{n\pi}\left(\sin(n\theta)\Big|_{\pi}^{2\pi}\right) = 0.$$

Now we compute k_n.

$$k_n = \frac{1}{\pi}\int_{\pi}^{2\pi} \sin(n\theta)d\theta = \frac{1}{n\theta}\left(-\cos(n\theta)\Big|_{\pi}^{2\pi}\right) = \frac{1}{n\pi}(\cos(n\pi) - 1) = \begin{cases} 0 & n = 2m \\ \frac{-2}{n\pi} & n = 2m+1. \end{cases}$$