545 Winter 2004 Midterm Solutions

1. a) Each fixed $w = (a, b)$ in the first circle rotates the entire sphere (a diffeomorphism) and hence maps lines that are tangent to the longitudinal second circle isomorphically. This shows that the rank of F is at least one everywhere.

On the inverse image of the poles I claim the rank is exactly one, because the poles are fixed under the rotations. In other words, $S^1 \times \{pole\}$ collapses to a point and hence the kernel of F_* has dimension at least one there, proving the claim.

Finally, away from the inverse image of the poles I claim the rank is 2. This is because for each fixed $z = (c, d)$ in the second circle, with $d \neq \pm 1$, F simply embeds the first circle as a latitude. Hence the pushforward of a nonzero vector tangent to the first circle will be linearly independent of the longitudinal vector described above.

b) Note that F is the restriction of a smooth map $G : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by the same formula. This suggests considering the commutative diagram

$$\begin{tikzcd}
S^1 \times S^1 \arrow[r, "i"] \arrow[d, "F"'] & \mathbb{R}^2 \times \mathbb{R}^2 \arrow[d, "G"] \\
S^2 \arrow[r, "j"'] & \mathbb{R}^3
\end{tikzcd}$$

where i, j are the evident smooth embeddings. (Notice that I am not using anything from Chapter 8 (i.e., embedded submanifolds) here, although in the future we should think of it that way.)

Since j_* is injective everywhere, we have

$$rank\, F_* = rank\, (jF)_* = rank\, (Gi)_*$$

everywhere. To compute the rank of $(Gi)_*$, note that the Jacobian of G at (a, b, c, d) is the matrix A:

$$\begin{pmatrix} 0 & -c & -b & 0 \\ c & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the image of i_* at (a, b, c, d) is spanned by $v_1 = (-b, a, 0, 0)$ and $v_2 = (0, 0, -d, c)$. Hence the image of $(Gi)_*$ is spanned by $Av_1 = (-ac, -bc, 0)$ and $Av_2 = (bd, -ad, c)$. If $c \neq 0$ these two vectors are linearly independent. If $c = 0$, then $Av_1 = 0$ but $Av_2 \neq 0$. Hence the rank of F_* is 1 on $S^1 \times \{(0, \pm 1)\}$ (i.e., the inverse image of the poles) and is 2 everywhere else.

Remark: Another good way was to use the exponential map $\mathbb{R} \times \mathbb{R} \longrightarrow S^1 \times S^1$. Then you only have to compute the rank of the composite

$$\mathbb{R} \times \mathbb{R} \longrightarrow S^1 \times S^1 \longrightarrow S^2 \longrightarrow S^3,$$

which is a pure Jacobian computation. Using hemisphere or stereographic coordinates for the source and target certainly works, but it is a very tedious approach. Always, always, strive for simplicity!

c) Let U denote the complement of the poles in S^2. Then by the calculation above, $F^{-1}U \longrightarrow U$ has rank 2 everywhere and hence is a local diffeomorphism by the inverse function theorem. Since F has rank one on the inverse image of the poles, U is maximal as required.

2. a) The idea of the proof will be to show that g is constant on fibres; then the "passing to the quotient" Prop. 7.18 can be applied. However, I'm going to arrange the proof in such a way as to illustrate (i) a couple of conceptual points; and (ii) the advantage (so I claim) of organizing your proof into lemmas or steps. Since I don't want to assume anything from chapter 8, I will not use the fact that the fibres are embedded submanifolds. See below for further comments on that.

The first conceptual point is one we've discussed several times. Consider the assertion

A. $Xg \equiv 0$ for every vertical vector field X.

Here we are requiring the vector field X to be globally defined. On the other hand, we could define a local vector field on M to consist of a pair (U, Y) where U is an open subset of M and Y is a smooth vector field defined on U. Then we have the alternative version of **A**:

B. $Yg \equiv 0$ on U for every vertical local vector field Y as above.

This brings us to our first lemma.

Lemma 0.1 *Assertions* **A** *and* **B** *are equivalent.*

Proof: **A** $\Rightarrow$ **B**: Suppose Y is a vertical local vector field defined on U. We need to show that for all $p \in U$, $Y_p g = 0$. Choose a bump function ψ that is identically one near p and has support in U. Then as usual the formula $X = \psi Y$ defines a smooth global vector field on M (zero outside U), and X is vertical. Hence $Y_p g = X_p g = 0$.

The converse is trivial, and in fact holds under the weaker hypothesis where condition **B** is only assumed for a collection of particular local vector fields whose domains cover M.

Now we're going to use the rank theorem (even though this means implicitly using it twice, since Prop. 7.18 depends on it). So we'll want to say something like "locally F looks like a projection from a product of Euclidean spaces", and then draw some conclusions. This suggests our second lemma. Suppose we are given a commutative diagram

$$\begin{tikzcd}
M_1 \arrow[r, "\alpha"] \arrow[d, "F_1"'] & M_2 \arrow[d, "F_2"] \\
N_1 \arrow[r, "\beta"'] & N_2
\end{tikzcd}$$

in which the vertical maps are surjective submersions and the horizontal maps are diffeomorphisms. In other words, F_1, F_2 are "isomorphic maps". Then we will need to know that the theorem we're trying to prove is invariant under such "isomorphisms".

Lemma 0.2 *With the notation above,*

(a) A vector field X on M_1 is vertical if and only $\alpha_* X$ is vertical. (Recall that pushforward along diffeomorphisms is defined for vector fields.)

(b) A smooth function g on M_1 is constant on the fibres of F_1 if and only if $g \circ \alpha^{-1}$ is constant on the fibres of F_2.

(c) $Xg \equiv 0$ for all vertical vector fields X on M_1 if and only if $Y(g \circ \alpha^{-1}) \equiv 0$ for all vertical vector fields Y on M_2.

The proof is just an unwinding of definitions. Our final lemma concerns the Euclidean projection case.

Lemma 0.3 *The theorem holds for the projection on the first n coordinates $\mathbb{R}^m \longrightarrow \mathbb{R}^n$.*

Proof: Suppose $Xg \equiv 0$ for all vertical vector fields X. The partial derivatives with respect to $x_{n+1}, ..., x_m$ are all vertical, so g is a function of $x_1, ..., x_n$ only. In other words, g descends as required.

We now prove the general theorem. Suppose $Xg \equiv 0$ for all vertical vector fields X on M. By Prop. 7.18 it is enough to show that g is constant on fibres. Since the fibres are connected, it is enough to show that g is locally constant on fibres. By the rank theorem, F is locally isomorphic to a Euclidean projection; more precisely, for any point $p \in M$ there exist neighborhoods U of p and V of $F(p)$ and a commutative diagram

$$\begin{tikzcd}
U \arrow[r, "\alpha"] \arrow[d, "F|_U"'] & \mathbb{R}^m \arrow[d, "\pi"] \\
V \arrow[r, "\beta"'] & \mathbb{R}^n
\end{tikzcd}$$

in which the horizontal maps are diffeomorphisms and π is projection on the first n coordinates. By Lemma 0.1, $Yg \equiv 0$ for every vertical vertical field Y on U. By Lemma 0.2, the analogous statement holds on $\mathbb{R}^m$; hence by Lemma 0.3, $g \circ \alpha^{-1}$ is constant on the fibres of π. Then by Lemma 0.2 again, g is constant on the fibres of $F\,|_U$. Hence g is locally constant on the fibres of F, as required.

Remark: If we assume the results of Chapter 8, then we know that the fibres are embedded submanifolds of M. Fix such a fibre and call it L. Then g restricts to a smooth function on L, and we want to show this restricted function is constant. Since L is connected, it suffices to show it is locally constant. Naturally, we'd like to do this by showing that for every smooth vector field Y on L, $Y(g\,|_L) \equiv 0$ (cf. earlier homework problem). To apply the hypothesis we would need to know that Y extends to a vertical vector field on M; then we'd be done. This is essentially Problem 8.11b, taking care to ensure the extended vector field is vertical.

2b. If F is a local diffeomorphism, then the only vertical vector field is the zero vector field. Hence every g satisfies the hypothesis, but clearly not every g can descend. For example, when F is the exponential map from the reals to the circle, only the periodic functions descend. An even simpler example was suggested by a couple of people: Let $M = \mathbb{R} \coprod \mathbb{R}$, $N = \mathbb{R}$, F the function which is the identity on each component of M, and let g be identically zero on one component and identically one on the other.

While I was typing this it occurred to me that there is a simpler counterexample yet, although it would probably get you arrested in most states. Take M to be two points, N to be one point, F the unique map, and g as in the preceeding example. In this case the only vector field is the zero vector field, and only the constant functions descend!

2c. Suppose $g = h \circ F$ and X is vertical. Then for all $p \in M$,

$$X_p g = X_p(h \circ F) = (F_{*p} X)h = 0,$$

where the middle equality is by definition of the pushforward.