Math 308C Midterm II – Solutions

1. Let A be a (3×3) matrix such that

$$A\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\3\\2\end{bmatrix}, \quad A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \quad \text{and } A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix},$$

where $\mathbf{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ is an unknown vector in $\mathbb{R}^3$.

For each of the following, determine whether there exists a vector $\mathbf{b} \in \mathbb{R}^\mathbf{3}$ such that A will have the given rank and nullity. If so, give an example of such a $\mathbf{b}$ (with actual numbers) – no explanation required. If not, briefly explain why not.

a. (3 pts.) rank $= 3$ and nullity $= 0$?

No such $\mathbf{b}$. *If the nullity of* A *were* 0, *then* $\mathcal{N}(A) = \{\mathbf{0}\}$. *But since* $A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, *we see that* $\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \in \mathcal{N}(A)$, *a contradiction. So the nullity of* A *must be at least 1.*

b. (3 pts.) rank $= 2$ and nullity $= 1$?

Yes, such a $\mathbf{b}$ *exists. The easiest way to see this is to consider* $\mathcal{R}(A)$. *Since* $A\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\3\\2\end{bmatrix}$, *we have* $\begin{bmatrix}-1 & 3 & 2\end{bmatrix}^T \in \mathcal{R}(A)$ *by definition, so the rank of* A *is at least 1. And since* $\mathbf{b} \in \mathcal{R}(A)$ *automatically, if we choose a vector* $\mathbf{b}$ *which is linearly independent from* $\begin{bmatrix}-1 & 3 & 2\end{bmatrix}^T$ *(i.e. not a scalar multiple of it), then the dimension of* $\mathcal{R}(A)$ *will be at least 2. Since the nullity of* A *is at least 1 (see part (a)), the rank-nullity theorem will then tell us that the rank of* A *is exactly 2 since* $n = 3$. *Thus we can choose any* $\mathbf{b}$ *that is not a scalar multiple of* $\begin{bmatrix}-1 & 3 & 2\end{bmatrix}^T$ *to get rank* $= 2$ *and nullity* $= 1$, *e.g.* $\mathbf{b} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ *or* $\mathbf{b} = \begin{bmatrix}1\\5\\18\end{bmatrix}$, *etc.*

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c. (3 pts.) rank = 1 and nullity = 2?

Yes, such a $\mathbf{b}$ *exists. This time we can either revisit our reasoning in part (b) or just consider* $\mathcal{N}(A)$. *Our reasoning in part (b) suggested that if we* did *choose a* $\mathbf{b}$ *that was a scalar multiple of* $\begin{bmatrix} -1 & 3 & 2 \end{bmatrix}^T$, *we would only end up with a rank of 1, since then* $\{\begin{bmatrix} -1 & 3 & 2 \end{bmatrix}^T\}$ *would be a basis for all of* $\mathcal{R}(A)$. *This suggests that here we should choose* $\mathbf{b} = c \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ *for some* $c \in \mathbb{R}$.

Alternately, considering $\mathcal{N}(A)$, *we know already that* $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \in \mathcal{N}(A)$ *(see (a)), so if we can force* $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \in \mathcal{N}(A)$ *as well, then we will have two linearly independent vectors in* $\mathcal{N}(A)$ *and hence a nullity of at least 2. Since we already know that the rank of A is at least 1 (see (b)), the rank-nullity theorem will then tell us that the nullity of A is exactly 2. This suggests choosing* $\mathbf{b} = \mathbf{0}$.

Either line of reasoning is correct; correct answers include

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -15 \\ 45 \\ 30 \end{bmatrix}, \textit{ etc.}$$

d. (3 pts.) rank = 1 and nullity = 1? (HINT: this one uses different reasoning than parts a–c.)

No such $\mathbf{b}$. *If there were such a* $\mathbf{b}$, *the rank-nullity theorem would say that, since A is* (3×3), *we must have* $\text{rank}(A)+\text{nullity}(A) = n = 3$. But of course $1 + 1 \neq 3$.

2. Short proofs:

a. (5 pts.) Suppose W is a 1-dimensional subspace of $\mathbb{R}^n$. Prove rigorously that any two nonzero vectors in W are scalar multiples of each other, that is, for any $\mathbf{v}$, $\mathbf{w} \in W$ such that $\mathbf{v}$, $\mathbf{w} \neq \mathbf{0}$, there is a $c \in \mathbb{R}$ such that $\mathbf{v} = c\mathbf{w}$.

Since $\dim(W) = 1$, *any basis for* W *must contain exactly one nonzero vector. Let* $\{\mathbf{u}\}$ *be such a basis. Then if* $\mathbf{v}$, $\mathbf{w} \in W$, *by the definition of a basis, we must have* $\mathbf{v} = a\mathbf{u}$ *and* $\mathbf{w} = b\mathbf{u}$ *for some* $a, b \in \mathbb{R}$. *If we assume that* $\mathbf{v}$, $\mathbf{w} \neq 0$, *then* $a, b \neq 0$. *Then dividing by* a *and* b, *respectively, we have* $\frac{1}{a}\mathbf{v} = \mathbf{u}$ *and* $\frac{1}{b}\mathbf{w} = \mathbf{u}$, *and so* $\frac{1}{a}\mathbf{v} = \frac{1}{b}\mathbf{w}$. *Thus* $\mathbf{v} = c\mathbf{w}$ *where* $c = \frac{a}{b}$. $\square$

b. (5 pts.) Suppose V and W are both subspaces of $\mathbb{R}^m$, and define the set $V + W$ to be

$$V + W = \{\mathbf{u} \in \mathbb{R}^m \mid \mathbf{u} = \mathbf{v} + \mathbf{w} \text{ for some } \mathbf{v} \in V \text{ and } \mathbf{w} \in W\}.$$

Show that $V + W$ is a subspace of $\mathbb{R}^m$.

To determine whether V *is a subspace or not, we must check the three criteria for subspaces:*

1. Is the zero vector in $V + W$?

Yes: since V *is a subspace of* $\mathbb{R}^m$, *we know* $\mathbf{0} \in V$, *and since* W *is also a subspace of* $\mathbb{R}^m$, $\mathbf{0} \in W$. *Thus* $\mathbf{0} = \mathbf{0} + \mathbf{0} \in V + W$.

2. If $\mathbf{u}_1$ *and* $\mathbf{u}_2 \in V + W$, *is* $\mathbf{u}_1 + \mathbf{u}_2 \in V + W$?

Yes: if $\mathbf{u}_1$, $\mathbf{u}_2 \in V + W$, *then by the definition of* $V + W$, *there must exist* $\mathbf{v}_1$, $\mathbf{v}_2 \in V$ *and* $\mathbf{w}_1$, $\mathbf{w}_2 \in W$ *such that* $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{w}_1$ *and* $\mathbf{u}_2 = \mathbf{v}_2 + \mathbf{w}_2$. *Then*

$$\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{v}_1 + \mathbf{w}_1 + \mathbf{v}_2 + \mathbf{w}_2 = (\mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{w}_1 + \mathbf{w}_2),$$

and since V *and* W *are subspaces of* $\mathbb{R}^m$, *we must have* $\mathbf{v}_1 + \mathbf{v}_2 \in V$ *and* $\mathbf{w}_1 + \mathbf{w}_2 \in W$, *respectively. Thus* $\mathbf{u}_1 + \mathbf{u}_2$ *is of the correct form and hence is in* $V + W$.

3. If $\mathbf{u} \in V + W$, *is* $a\mathbf{u} \in V + W$ *for all* $a \in \mathbb{R}$?

Yes: if $\mathbf{u} \in V + W$, *then* $\mathbf{u} = \mathbf{v} + \mathbf{w}$ *for some* $\mathbf{v} \in V$ *and* $\mathbf{w} \in W$, *and again, since* V *and* W *are subspaces of* $\mathbb{R}^m$, *for any* $a \in \mathbb{R}$, *we have* $a\mathbf{v} \in V$ *and* $a\mathbf{w} \in W$. *Thus* $a\mathbf{u} = a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$ *is of the correct form and hence is in* $V + W$.

Thus V *is indeed a subspace of* $\mathbb{R}^m$. $\square$

3. Let W be the subspace given by $W = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}\right)$.

a. (4 pts.) Use the Gram-Schmidt algorithm to find an orthonormal basis for W.

Set $\mathbf{w}_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ *and* $\mathbf{w}_2 = \begin{bmatrix} -4 & 0 & 1 \end{bmatrix}^T$. *Applying the Gram-Schmidt algorithm, we first construct an orthogonal basis,* $\{\mathbf{u}_1, \mathbf{u}_2\}$*:*

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \mathbf{u}_2 &= \mathbf{w}_2 - \frac{\mathbf{w}_2^T \mathbf{u}_1}{\mathbf{u}_1^T \mathbf{u}_1} \mathbf{u}_1 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} - \frac{(-4)}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}. \end{aligned}$$

Now we need to normalize our basis vectors to obtain an orthonormal basis, $\{\mathbf{v}_1, \mathbf{v}_2\}$*:*

$$\begin{aligned} \mathbf{v}_1 &= \frac{\mathbf{u}_1}{||\mathbf{u}_1||} = \frac{1}{\sqrt{1^2 + 1^2 + 0^2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \\ \mathbf{v}_2 &= \frac{\mathbf{u}_2}{||\mathbf{u}_2||} = \frac{1}{\sqrt{(-2)^2 + 2^2 + 1^2}} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}. \end{aligned}$$

b. (4 pts.) Find coordinates for the vector $\begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$ with respect to the basis you found in part (b).

Let $\mathbf{x} = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$. *Since the basis we found in part (a) is orthonormal, if* $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$, *we may simply compute*

$$\begin{aligned} a_1 &= \mathbf{x} \cdot \mathbf{v}_1 = 2\left(\frac{1}{\sqrt{2}}\right) + 6\left(\frac{1}{\sqrt{2}}\right) + 1\,(0) = \frac{8}{\sqrt{2}} = 4\sqrt{2}, \\ a_2 &= \mathbf{x} \cdot \mathbf{v}_2 = 2\left(-\frac{2}{3}\right) + 6\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right) = \frac{9}{3} = 3. \end{aligned}$$

So the coordinates of $\mathbf{x}$ *with respect to* $\{\mathbf{v}_1, \mathbf{v}_2\}$ *are* $(4\sqrt{2}, 3)$.

4. Suppose C is a (4×4) matrix whose augmented matrix $[\,C\,|\,\mathbf{b}\,]$ is row-equivalent to

$$\left[\begin{array}{cccc c} 1 & -3 & 0 & 0 & 7b_3 - b_4 \\ 0 & 0 & 1 & 0 & 4b_1 \\ 0 & 0 & 0 & 1 & -3b_2 + b_4 \\ 0 & 0 & 0 & 0 & 2b_1 - 4b_2 - b_3 \end{array}\right].$$

a. (4 pts.) Find a basis for $\mathcal{R}(C)$.

From the bottom row of the row-reduced $[\,C\,|\,\mathbf{b}\,]$, *we can see that the system* $C\mathbf{x} = \mathbf{b}$ *will be consistent (and hence* $\mathbf{b} \in \mathcal{R}(C)$*) if* $2b_1 - 4b_2 - b_3 = 0$. *Solving for* b_3, *we get* $b_3 = 2b_1 - 4b_2$. *Thus the general form of a vector in* $\mathcal{R}(C)$ *is*

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ 2b_1 - 4b_2 \\ b_4 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ -4 \\ 0 \end{bmatrix} + b_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and so a basis for $\mathcal{R}(C)$ *is* $\left\{ \begin{bmatrix} 1 & 0 & 2 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & -4 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \right\}$.

b. (4 pts.) Find a basis for $\mathcal{N}(C)$. (HINT: set $b_1 = b_2 = b_3 = b_4 = 0$.)

Setting $b_1 = b_2 = b_3 = b_4 = 0$, *we can read off the solutions* $\mathbf{x}$ *to the homogeneous system* $C\mathbf{x} = \mathbf{0}$ *— which are precisely the elements of* $\mathcal{N}(C)$ *— from the above matrix:*

$$x_1 = 3x_2, \;\; x_3 = 0, \;\; \textit{and } x_4 = 0.$$

Thus the general form of a vector $\mathbf{x} \in \mathcal{N}(C)$ *is*

$$\mathbf{x} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and so a basis for $\mathcal{N}(C)$ *is* $\left\{ \begin{bmatrix} 3 & 1 & 0 & 0 \end{bmatrix}^T \right\}$.

c. (2 pts.) Find a basis for the row space of C.

Ignoring the augmented column of the row-reduced $[\,C\,|\,\mathbf{b}\,]$, *we may simply read off its non-zero rows, since they form a basis for the row space of* C:

$$\left\{ \begin{bmatrix} 1 & -3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right\}.$$

5. True or False: circle one (2 pts each)

a. [✓ T] F If $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$ and $\mathbf{v}$ is a nonzero vector in $\mathbb{R}^n$ such that $\mathbf{v}^T\mathbf{w}_i = 0$ for $i = 1, 2, 3$, then $n \geq 4$. (HINT: consider the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{v}\}$.)

Since the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ *are pairwise orthogonal and* $\mathbf{v}$ *is orthogonal to each of them,* $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{v}\}$ *is also an orthogonal set. Furthermore, all of these vectors are nonzero by hypothesis, and we know that any orthogonal set of nonzero vectors is linearly independent. If* n *were strictly less than* 4*, any set of four vectors would have to be linearly dependent, a contradiction. So we must have* $n \geq 4$.

b. [✓ T] F For any $(m \times n)$ matrix A, the map $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation.

This was stated as a fact in class. Alternately, using the properties of matrix multiplication, we can easily check that the given map T *satisfies the two properties required for linear transformations:*

$$T(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y}), \text{ and}$$
$$T(c\mathbf{x}) = A(c\mathbf{x}) = c(A\mathbf{x}) = cT(\mathbf{x}) \text{ for any } c \in \mathbb{R}.$$

c. T [✓ F] If W is a subspace of dimension k, then any spanning set for W contains at most k vectors.

Spanning sets need not be minimal. Thus, although every minimal spanning set (i.e., basis) for W *must contain exactly* k *vectors, an arbitrary spanning set for* W *must contain* **at least** k *vectors, and in particular, it could contain more.*

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d. T [✓ F] If $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, then $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, *using the properties of linear transformations, we compute that*

$$\begin{aligned} T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) &= 2\cdot T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 3\cdot T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 0 \end{bmatrix}. \end{aligned}$$

e. [✓ T] F If the columns of an $(n \times n)$ matrix C span $\mathbb{R}^n$, then C must be nonsingular.

$\mathbb{R}^n$ *has dimension* n, *so since the* n *columns of* C *span* $\mathbb{R}^n$, *they must in fact comprise a basis for* $\mathbb{R}^n$, *and hence they must be linearly independent. If the columns of a square matrix are linearly independent, then the matrix is nonsingular.*