

# Math 308E final exam review questions

## I Solving systems of linear equations using matrices

- i. What is a system of linear equations? What is a solution to such a system?
- ii. How many solutions can a system have?
- iii. What is an  $(m \times n)$  matrix?
- iv. What is the coefficient matrix of a system? What is the augmented matrix?
- v. When are two systems equivalent? When are two matrices row equivalent?
- vi. What are the elementary operations for systems? Elementary row operations for matrices?
- vii. What is the algorithm for Gauss-Jordan elimination?
- viii. What is echelon form for a matrix? Reduced echelon form?
- ix. What is a consistent system? An inconsistent system?
- x. What is a homogeneous system?
- xi. What is the trivial solution to a homogeneous system? What is a non-trivial solution?
- xii. How many solutions can a homogeneous system have if it has  $m$  equations in  $n$  unknowns and  $m < n$ ? If  $m = n$ ? If  $m > n$ ?

## II Matrix and vector arithmetic

- i. When are two matrices equal?
- ii. How do you add two matrices?
- iii. How do you multiply a scalar and a matrix (scalar multiplication)?
- iv. How do you multiply two matrices? What condition on the matrices' dimensions is required in order to be able to multiply them?
- v. How is the dot product defined in terms of matrix multiplication? How is the length  $\|\mathbf{x}\|$  of a vector  $\mathbf{x}$  defined?
- vi. What are the properties of matrix addition, multiplication, and the transpose operation with respect to each other?

## III Linear independence, nonsingularity

- i. What is the definition of linear independence for a set of vectors? Linear dependence?
- ii. What is a linear combination of vectors?
- iii. What are the standard basis vectors for  $\mathbb{R}^n$ ,  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ ?
- iv. What is the definition of a nonsingular matrix? A singular matrix?
- v. How is the notion of nonsingularity for matrices related to the notion of linear independence for sets of vectors?
- vi. What can you say about solutions of  $A\mathbf{x} = \mathbf{b}$  when  $A$  is nonsingular? When  $A$  is singular?
- vii. When is a matrix invertible? What is an inverse for a matrix?
- viii. How can you use the inverse of a matrix  $A$  to solve the linear system  $A\mathbf{x} = \mathbf{b}$ ?

- ix. How can you calculate the inverse of a  $(2 \times 2)$  matrix? Of an  $(n \times n)$  matrix?
- x. What are the properties of matrix inverses with respect to addition, scalar multiplication, matrix multiplication, and transposes?
- xi. Can you list at least *seven* things that are equivalent to the statement that a matrix  $A$  is nonsingular?

#### IV Subspaces, spanning sets, bases, dimension

- i. What is a subspace of  $\mathbb{R}^n$ ? What properties must you check to see whether a subset  $W \subset \mathbb{R}^n$  is a subspace?
- ii. What do subspaces of  $\mathbb{R}^3$  correspond to geometrically?
- iii. What is the span of a set of vectors,  $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$ ? Is it subspace?
- iv. Given a subspace  $W$ , what is a spanning set for  $W$ ? What is a basis for  $W$ ?
- v. Given a basis for a subspace  $W$ , in how many ways can an arbitrary vector  $\mathbf{x} \in W$  be written as a linear combination of basis vectors? Why?
- vi. Given a matrix  $A$ , what is its null space,  $\mathcal{N}(A)$ ? How would you find a basis for  $\mathcal{N}(A)$ ?
- vii. Given a matrix  $A$ , what is its range space,  $\mathcal{R}(A)$ ? How would you find a basis for  $\mathcal{R}(A)$ ? (There are at least three distinct methods.)
- viii. Given a matrix  $A$ , what is its row space,  $\text{RowSp}(A)$ ? How would you find a basis for  $\text{RowSp}(A)$ ?
- ix. What is the null space of any nonsingular matrix? What is its range?
- x. Given a subspace  $W$  of the form  $W = \{\mathbf{x} : \mathbf{x} \text{ satisfies some system of linear equations}\}$ , how can you find a basis for  $W$ ?
- xi. Given a subspace  $W$ , what is its dimension?
- xii. Given a matrix  $A$ , what is  $\text{rank}(A)$ ? What is  $\text{nullity}(A)$ ? How are the two related?
- xiii. Given a matrix  $A$ , how are  $\text{rank}(A)$  and  $\text{rank}(A^T)$  related?
- xiv. Given a subspace  $W$  with  $\dim(W) = p$ , what do you know about sets of vectors in  $W$  containing  $p + 1$  or more vectors? Containing fewer than  $p$  vectors? Containing  $p$  linearly independent vectors? Containing  $p$  vectors which span  $W$ ?
- xv. If  $U$  and  $V$  are subspaces and  $U \subset V$ , how are  $\dim(U)$  and  $\dim(V)$  related? If  $\dim(U) = \dim(V)$ , what does that imply about  $U$  and  $V$ , if anything?

#### V Orthogonal and orthonormal sets of vectors, coordinates

- i. What does it mean for two vectors to be orthogonal? What does it mean for a *set* of vectors to be orthogonal?
- ii. If  $S$  is an orthogonal set of vectors, is it necessarily linearly independent? What if  $S$  is an orthogonal set of *nonzero* vectors?
- iii. What is an orthogonal basis for a subspace  $W$ ? What is an orthonormal basis for  $W$ ?

- iv. Given a linearly independent set of vectors, how would you obtain an orthogonal set with the same span?
- v. Given an orthogonal set of nonzero vectors, how would you obtain an orthonormal set with the same span?
- vi. What are the *coordinates* of a vector  $\mathbf{v}$  with respect to a basis  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ ? If the basis is orthogonal, how can you easily compute these coordinates? If it's orthonormal?

## VI Linear transformations

- i. What is the definition of a linear transformation? Given two subspaces  $V$  and  $W$  and a function  $T : V \rightarrow W$  between them, how would you determine whether or not  $T$  is a linear transformation?
- ii. What is the null space  $\mathcal{N}(T)$  of a linear transformation  $T$ ? What is the range space  $\mathcal{R}(T)$ ? What are the rank and nullity of  $T$ ?
- iii. Is it necessarily true that for any linear transformation  $T$ , there is some matrix  $A$  such that  $T$  is given by  $T(\mathbf{x}) = A\mathbf{x}$  for all vectors  $\mathbf{x}$ ?
- iv. Suppose you know that  $T(\mathbf{x}) = A\mathbf{x}$  for some  $(m \times n)$  matrix  $A$ . Given  $T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)$ , how can you find  $A$ ? Given a basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  for  $\mathbb{R}^n$  and  $T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)$ , how can you find  $A$ ?
- v. What is an orthogonal transformation on  $\mathbb{R}^2$ ? What is the matrix representation of a rotation through an angle of  $\theta$  on  $\mathbb{R}^2$ ?
- vi. What property must an  $(n \times n)$  matrix satisfy in order to be called an orthogonal matrix? What do you know about the columns of such a matrix?

## VII Determinants, eigenvalues, eigenvectors, similarity, diagonalization

- i. How would you compute the determinant of a  $(2 \times 2)$  matrix? A  $(3 \times 3)$  matrix?
- ii. What are the properties of determinants with regard to matrix multiplication? Scalar multiplication? Matrix inverses?
- iii. What can you say about  $A$  if  $\det(A) = 0$ ? If  $\det(A) \neq 0$ ?
- iv. What is an eigenvalue for a square matrix  $A$ ?
- v. What is the characteristic polynomial of a square matrix  $A$ ? What is its relation to the eigenvalues of  $A$ ?
- vi. How are the eigenvalues of  $A$  related to those of  $A^k$  for  $k = 2, 3, \dots$ ?  $A^{-1}$  (if it exists)?  $A^T$ ?
- vii. What can you say about the eigenvalues of a singular matrix? A nonsingular matrix?
- viii. What is a diagonal matrix? How can you *quickly* determine the eigenvalues of a diagonal matrix?
- ix. What is an upper triangular matrix? How can you *quickly* determine the eigenvalues of an upper triangular matrix?

- x. What is an eigenvector corresponding to a given eigenvalue? Given an eigenvalue, how can you find the corresponding eigenvectors?
- xi. What is an eigenspace? Is it a subspace?
- xii. If  $A$  is the  $(2 \times 2)$  matrix corresponding to a linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , what is the geometric meaning of an eigenvector  $\mathbf{x}$  for  $A$ ? An eigenspace? An eigenvalue?
- xiii. What is the algebraic multiplicity of an eigenvalue? The geometric multiplicity?
- xiv. When is a matrix defective?
- xv. If a matrix has distinct eigenvalues (i.e. each has algebraic multiplicity 1), what do you know about the linear (in)dependence of its eigenvectors?
- xvi. What does it mean for two  $(n \times n)$  matrices to be similar? How are their determinants related? Their characteristic polynomials? Their eigenvalues?
- xvii. What does it mean for a matrix  $A$  to be diagonalizable? How are diagonalizability and defectiveness related?
- xviii. Given a diagonalizable matrix  $A$ , how would you find a matrix  $S$  which diagonalizes it? Is  $S$  necessarily unique?
- xix. How does the process of diagonalization help you to compute  $A^N$  for large integers  $N$ ?

## VIII Applications: data fitting, initial value problems, least-squares, difference equations

- i. Given a set of  $n$  points in the plane, how do you find a polynomial which fits them exactly? What degree polynomial is necessary?
- ii. How can you use a system of linear equations to solve the following type of initial value problem?  
 Given  $t_0, t_1, \dots, t_n$  and  $y_0, y_1, \dots, y_n \in \mathbb{R}$ , find constants  $a_0, a_1, \dots, a_n$  such that the function  $y(x) = a_0 e^{t_0 x} + a_1 e^{t_1 x} + \dots + a_n e^{t_n x}$  satisfies the constraints  $y(0) = y_0, y'(0) = y_1, \dots, y^{(n)}(0) = y_n$ .
- iii. What is a least-squares solution to a linear system  $A\mathbf{x} = \mathbf{b}$ ? What do you know about the least squares solution if the linear system is inconsistent? If it is consistent?
- iv. How can you find the least-squares linear fit to a given data set? What about a least-squares quadratic fit?
- v. Given a difference equation of form  $\mathbf{x}_k = A\mathbf{x}_{k-1}$  where  $A$  is an  $(n \times n)$  matrix and  $\mathbf{x}_0 \in \mathbb{R}^n$  is given, how can you find a formula for  $\mathbf{x}_k$  in terms of  $\mathbf{x}_0$  and  $k$  only? What condition on  $A$  is required?
- vi. Given a system of differential equations of the form  $\mathbf{x}'(t) = A\mathbf{x}(t)$  for some square matrix  $A$ , what is the most general solution  $\mathbf{x}(t)$ ? What condition on  $A$  is required?
- vii. How would you find the particular solution of the system  $\mathbf{x}'(t) = A\mathbf{x}(t)$  that satisfies the initial value  $\mathbf{x}(0) = \mathbf{x}_0$ , where  $\mathbf{x}_0$  is some specified vector in  $\mathbb{R}^2$ ?