Name: ______________________________

Math 308D Midterm I

30 January 2006

Instructions:

- This is a closed book exam. Graphing calculators are not allowed.
- You may use one $8\frac{1}{2}$" × 11" sheet of notes.
- Show your work! If you don't show your work, I won't be able to give you partial credit.
- Use the backs of pages if you run out of space.
- Good luck!

Problem 1	9 points	
Problem 2	12 points	
Problem 3	10 points	
Problem 4	6 points	
Problem 5	13 points	
Total	50 points	

1. (9 points) The following matrices were obtained by taking the *augmented matrix* $[A|\mathbf{b}]$ of a linear system and reducing it to (reduced) row echelon form. In each case, tell me how many solutions the original system had, and explain with (just) a few words.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

2. (12 points) Suppose B is the 3×3 matrix given by

$$B = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}.$$

(a) (6 points) Find B^{-1}, the inverse of B.

(b) (4 points) Using your answer to part (a), solve the equation $B\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

(c) (2 points) Without making any calculations, determine how many solutions there are to $B\mathbf{x} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$. *Briefly* explain how you knew that.

3. (10 points) True/False: Mark a 'T' or 'F' beside each statement. *No explanation is necessary.*

(a) If A is a nonsingular $n \times n$ matrix with $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$, then A is necessarily the identity I_n.

(b) If P, Q, and R are nonsingular $(n \times n)$ matrices such that $PQR = I_n$, then $Q^{-1} = PR$.

(c) The collection $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \end{bmatrix} \right\}$ forms a linearly dependent set of vectors in $\mathbb{R}^2$.

(d) Any linear system of 3 equations in 4 unknowns necessarily has at least one solution.

(e) If A and B are both nonsingular $(n \times n)$ matrices, then so is AB.

4. (6 points) Suppose C is a 3×2 matrix, and we know that $C \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $C \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

(a) (3 points) What is $C \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

(b) (3 points) How many solutions are there to $C\mathbf{x} = \mathbf{0}$? (explain briefly)

5. (13 points) Suppose D is the following matrix:

$$D = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -3 & 1 & -3 \end{bmatrix}.$$

(a) (6 points) Show that the columns of D form a linearly *dependent* set of vectors.

(b) (2 points) What is the rank of the matrix D?

(c) (5 points) Find a condition on the vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that the equation $D\mathbf{x} = \mathbf{b}$ is *consistent* (has a solution). Your answer should be some relationship between b_1, b_2, and b_3.

EXTRA CREDIT (1 point): What does the word *matrix* mean in Latin?

Short proofs

6. Suppose A is an $n \times n$ matrix with the property that $A^2 = \mathscr{O}$ (where $\mathscr{O}$ is the *zero matrix*). Show that A is singular.

7. A nonsingular $n \times n$ matrix is called *orthogonal* if $A^{-1} = A^T$. Show that if A is an orthogonal matrix, then $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for any vector $\mathbf{x}$.