Objective

We wish to provide evidence for the main conjecture of Iwasawa theory formulated by Greenberg. Our result involving Selmer groups is motivated by a factorization formula obtained by Dasgupta (originally conjectured by Citro).

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4-dim. and 3-dim. Galois rep. arising from a Hida family \( \mathcal{F} \)

Notations:
- \( p \geq 5 \)
- \( \mathbb{Q}_\infty \) - the cyclotomic extension of \( \mathbb{Q} \) with \( \Gamma := \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong \mathbb{Z}_p \)
- \( \mathcal{F} \) - Hida family
- \( \mathcal{T}_\mathcal{F} \) - integral closure of the irreducible component of the ordinary Hecke algebra through \( \mathcal{F} \)

We make the following assumptions on the Hida family \( \mathcal{F} \):
- The residual representation \( \mathcal{T}_\mathcal{F} \) attached to \( \mathcal{F} \) is absolutely irreducible.
- The restriction \( \mathcal{T}_\mathcal{F} \big|_{\Gamma} \) to the inertia subgroup at \( p \) is \( p \)-distinguished.

We have a 4-dimensional representation:
\[
\rho_{\mathcal{F}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_4(\mathcal{T}_\mathcal{F}), \quad \text{where} \quad \mathcal{T}_\mathcal{F} = \mathcal{T}_\mathcal{F} \otimes \mathcal{T}_\mathcal{F}.
\]

- \( \theta_{\mathcal{F},\mathcal{F}} \in \text{Fr}(\mathcal{T}_\mathcal{F}[\Gamma]) \) : 3-variable (Rankin-Selberg) \( p \)-adic L-function constructed by Hida.

\( \mathcal{F} \) has 2 weight variables and 1 cyclotomic variable.

- \( \text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee \) - The dual of the Selmer group associated to \( \rho_{\mathcal{F}} \) is a finitely generated torsion module over \( \mathcal{T}_\mathcal{F}[\Gamma] \).

Under the natural map \( \pi : \mathcal{T}_\mathcal{F} \to \mathbb{Q}_p \), we have the following decomposition of Galois representations:
\[
\pi \circ \rho_{\mathcal{F}} \cong \text{Ad}^0(\rho_{\mathcal{F}}) \oplus \text{Trivial representation}.
\]

\( \text{Ad}^0(\rho_{\mathcal{F}}) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_4(\mathcal{T}_\mathcal{F}) \).

- \( \theta_{\mathcal{F},\mathcal{F}}(\pi) \in \text{Fr}(\mathcal{T}_\mathcal{F}[\Gamma]) \) : 2-variable \( p \)-adic L-function due to work of Coates-Schmidt and Hida. We have 1 weight variable and 1 cyclotomic variable.

- \( \text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee \) - The dual of the Selmer group is a finitely generated torsion module over \( \mathcal{T}_\mathcal{F}[\Gamma] \)

Divisors

Let \( \mathcal{T} \) be an integrally closed domain. Let \( M \) be a finitely generated torsion module over \( \mathcal{T} \).
- For each height 1 prime \( p \) in \( \mathcal{T} \), the localization \( \mathcal{T}_p \) is a DVR. By the structure theorem for DVRs
  \[
  M \otimes \mathcal{T}_p \cong \bigoplus_{n \geq 0} \mathcal{T}_p \left( \frac{T_p}{p^n} \right),
  \]
  where \( \pi \) is a uniformizer in \( \mathcal{T}_p \), \( \text{val}_p(M) = \sum a_i \).
- The divisor group \( \text{Div}(\mathcal{M}) \) is the free abelian group generated by height 1 primes of \( \mathcal{T} \).

\[
\text{Div}_\mathcal{P}(M) = \sum_{\text{primes } \mathcal{P} \text{ of } \mathcal{T}} \text{val}_\mathcal{P}(M) \cdot \mathcal{P} 
\]

Theorem on the analytic side [Dasgupta]

\[
\pi(\theta_{\mathcal{F},\mathcal{F}}) = \theta_{\mathcal{F},\mathcal{F}} \cdot \frac{\theta_{\mathcal{F},\mathcal{F}}}{\xi_1}
\]

Here, \( \theta_{\mathcal{F},\mathcal{F}} \in \mathcal{T}_\mathcal{F}[\Gamma] \) and \( \frac{\theta_{\mathcal{F},\mathcal{F}}}{\xi_1} \) is the generator for the Kubota-Leopoldt \( p \)-adic L-function.

Theorem on the algebraic side [P.]

We have the following equality of elements in the divisor group of \( \mathcal{T}_\mathcal{F}[\Gamma] \):
\[
\text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]}(\text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee \otimes \mathcal{T}_\mathcal{F}[\Gamma]) = \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]}(\text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee) + \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]}(\text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee) - \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]}(\mathcal{T}_\mathcal{F}[\Gamma]).
\]

Specialization result [P.]

Suppose \( \theta_{\mathcal{F},\mathcal{F}} \in \mathcal{T}_\mathcal{F}[\Gamma] \) and we have the following inequality in the divisor group of \( \mathcal{T}_\mathcal{F}[\Gamma] \):
\[
\text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]} \left( \mathcal{T}_\mathcal{F}[\Gamma] \right) \leq \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]} \left( \text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee \right).
\]

Then, we have \( \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]} \left( \mathcal{T}_\mathcal{F}[\Gamma] \right) \leq \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]} \left( \text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee \otimes \mathcal{T}_\mathcal{F}[\Gamma] \right) \) and
\[
\text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]} \left( \mathcal{T}_\mathcal{F}[\Gamma] \right) \left( \frac{\mathcal{T}_\mathcal{F}[\Gamma]}{\theta_{\mathcal{F},\mathcal{F}}} \right) = \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]} \left( \text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee \otimes \mathcal{T}_\mathcal{F}[\Gamma] \right) \left( \frac{\mathcal{T}_\mathcal{F}[\Gamma]}{\theta_{\mathcal{F},\mathcal{F}}} \right) = \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]} \left( \text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee \otimes \mathcal{T}_\mathcal{F}[\Gamma] \right).
\]

Remark: (3) can be expected to hold due to the Euler system machinery developed by Kings-Lei-Loeffler-Zerbes.

Heuristic

- (Mazur-Wiles) \( \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]} \left( \mathcal{T}_\mathcal{F}[\Gamma] \right) = \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]}(\text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee) \).
- (Hida-Tilouine-Urban) \( \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]} \left( \mathcal{T}_\mathcal{F}[\Gamma] \right) = \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]}(\text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee) \).
- Suppose (3) holds. Then
  \[
  \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]} \left( \mathcal{T}_\mathcal{F}[\Gamma] \right) + \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]} \left( \mathcal{T}_\mathcal{F}[\Gamma] \right) \left( \frac{\mathcal{T}_\mathcal{F}[\Gamma]}{\theta_{\mathcal{F},\mathcal{F}}} \right) \leq \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]}(\text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee) + \text{Div}_{\mathcal{T}_\mathcal{F}[\Gamma]}(\text{Sel}_{\mathcal{F}}(\mathbb{Q}_\infty)^\vee).
  \]

Combining these results with the specialization result, we then have the main conjecture for \( \mathcal{T}_\mathcal{F} \) over \( \mathbb{Q}_\infty \).