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Generic singularities of certain Schubert varieties

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Abstract. Let G be a connected semisimple algebraic group, B a Borel subgroup, T a maximal torus in B with Weyl group W, and Q a subgroup containing B. For $w \in W$, let X_{wQ} denote the Schubert variety $\overline{BwQ}/Q$. For $y \in W$ such that $X_{yQ} \subseteq X_{wQ}$, one knows that ByQ/Q admits a T-stable transversal in X_{wQ}, which we denote by $\mathcal{N}_{yQ,wQ}$. We prove that, under certain hypotheses, $\mathcal{N}_{yQ,wQ}$ is isomorphic to the orbit closure of a highest weight vector in a certain Weyl module. We also obtain a generalisation of this result under slightly weaker hypotheses. Further, we prove that our hypotheses are satisfied when Q is a maximal parabolic subgroup corresponding to a minuscule or cominuscule fundamental weight, and X_{yQ} is an irreducible component of the boundary of X_{wQ} (that is, the complement of the open orbit of the stabiliser in G of X_{wQ}). As a consequence, we describe the singularity of X_{wQ} along ByQ/Q and obtain that the boundary of X_{wQ} equals its singular locus.

Introduction

Let G be a connected semisimple algebraic group over k, an algebraically closed field of arbitrary characteristic. Choose a Borel subgroup B, a maximal torus T of B with Weyl group W, and a subgroup $Q \supseteq B$. For $w \in W$, let X_{wQ} denote the Schubert variety $\overline{BwQ}/Q$ in G/Q, and let $\mathrm{Bd}(X_{wQ})$ denote its boundary, that is, the complement of the open orbit of $\mathrm{Stab}_G(X_{wQ})$. For $y, w \in W$ such that $X_{yQ} \subseteq X_{wQ}$, it is well-known that the Bruhat cell $C_{yQ} := ByQ/Q$ admits a natural T-stable transversal in X_{wQ}, which we denote by $\mathcal{N}_{yQ,wQ}$ (see 1.2). In this paper we study, in certain cases, the singularity of X_{wQ} along C_{yQ}, that is, the singularity of $\mathcal{N}_{yQ,wQ}$ at the point yQ/Q. The most interesting case occurs when X_{yQ} is an irreducible component of the singular locus of X_{wQ}. Then the singularity of $\mathcal{N}_{yQ,wQ}$ at yQ/Q is isolated; it is the generic singularity of the title.

After some preliminaries in Sect. 1, we prove in Sect. 2 the main result of this paper (Theorem 2.6). It asserts that, under certain specific conditions on y and w, the T-variety $\mathcal{N}_{yQ,wQ}$ is isomorphic to the orbit closure of a highest weight vector in a certain Weyl module for a certain reductive subgroup containing T. As a consequence, we compute the Kazhdan-Lusztig polynomial $P_{y,w}$ (assuming y, w maximal in their W_Q-cosets) and the multiplicity of X_{wQ} along C_{yQ} (Corollary 2.7). In Sect. 3, we consider the case where Q is a maximal parabolic subgroup corresponding to a minuscule weight. We assume that G is simply-laced, which entails no loss of generality. Using a result of Lakshmibai-Weyman, which asserts that the Bruhat-Chevalley order in W/W_Q is generated by the simple reflections, we first show that for every irreducible component X_{yQ} of $\mathrm{Bd}(X_{wQ})$, the conditions of Sect. 2 are satisfied. Then, using Theorem 2.6, we deduce that $\mathrm{Bd}(X_{wQ})$ is exactly the singular locus of X_{wQ} and obtain a geometric description of the generic singularities of X_{wQ}.

Our description of the singular locus, and the value of generic multiplicities and Kazhdan-Lusztig polynomials, could be deduced from the case-by-case analysis given in [12], for classical groups, and from the computation of Kazhdan-Lusztig polynomials given in [1], for types E_6, E_7. In fact, these values are known, more generally, for all pairs of Schubert varieties $X_{yQ} \subseteq X_{wQ}$ in a minuscule G/Q [13], [1], [12]. But our description of generic singularities gives a more precise geometric information.

In Sect. 4, we begin by a generalisation of Theorem 2.6: for certain y and w, the T-variety $\mathcal{N}_{yQ,wQ}$ is isomorphic to a certain multicone in a direct sum of Weyl modules (Theorem 4.1). We then study the generic singularities of Schubert varieties in the variety of Lagrangian subspaces of a symplectic space k^{2n}. Again we find that the singular locus of each Schubert variety is its boundary, and, using Theorem 4.1, we give an explicit description of the transversals. As a consequence, formulae for the corresponding Kazhdan-Lusztig polynomials and multiplicities are obtained (the explicit formulae for the latter are perhaps new). Finally, we work out the case of Schubert varieties in a smooth quadric, or in the variety of flags of type $(1, n)$ in k^{n+1}, by elementary geometric arguments.

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1 Preliminaries

1.1

Throughout the paper, the base field k is algebraically closed and of arbitrary characteristic. Let G be a semisimple, connected and simply-connected, algebraic group over k. Let T be a maximal torus inside B, a Borel subgroup. Let U^- be the unipotent radical of B^-, the Borel subgroup such that $B^- \cap B = T$. Also, if Q is a parabolic subgroup containing B, let L_Q denote the Levi subgroup of Q containing T, let Q^- be the unique parabolic subgroup such that $Q^- \cap Q = L_Q$, and let U_Q^- be the unipotent radical of Q^-.

Let R be the root system of (G, T). For $\alpha \in R$, let U_α be the corresponding root subgroup, and let $U_\alpha^\times = U_\alpha \setminus \{1\}$. Let R^+ be the set of roots of T in $\mathrm{Lie}(B)$, let $R^- = -R^+$, and let $\varDelta$ be the set of simple roots in R^+. For a subset I of $\varDelta$, let P_I be the parabolic subgroup generated by B and the $U_{-\alpha}$, for $\alpha \in I$, and let $R_I = R \cap \mathbb{Z}I$, $R_I^\pm = R_I \cap R^\pm$. If $Q = P_I$, then L_Q, R_I, $R_I^\pm$ are also denoted by L_I, R_Q, $R_Q^\pm$, respectively.

1.2

Let $W = N_G(T)/T$ be the Weyl group and let $\ell(\cdot)$ (resp. $\leq$) denote the length function (resp. the Bruhat-Chevalley order) on W with respect to the set of simple reflections $\{s_\alpha,\ \alpha \in \varDelta\}$. For $I \subseteq \varDelta$, let W_I denote the subgroup of W generated by $\{s_\alpha,\ \alpha \in I\}$, let w_I denote the unique element of W_I such that $w_I(R_I^+) = R_I^-$, and let $W^I = \{w \in W \mid w(R_I^+) \subseteq R^-\}$, the set of maximal representatives of W/W_I.

For $w \in W$, let e_{wB} denote the point wB/B of G/B, let $C_{wB} = Be_{wB}$ be the B-orbit of e_{wB}, and let $X_{wB} = \overline{C_{wB}}$ be its Zariski closure. Recall that $\dim X_{wB} = \ell(w)$ and that $X_{yB} \subseteq X_{wB} \iff y \leq w$. More generally, let Q be a parabolic subgroup containing B. For $w \in W$, let e_{wQ} denote the point wQ/Q of G/Q, let $C_{wQ} = Be_{wQ}$, and let $X_{wQ} = \overline{C_{wQ}}$. Note that these depend only on the coset wW_Q, where $W_Q = \{w \in W \mid wQ = Q\}$. If $Q = P_I$, then $W_Q = W_I$ and we shall also write W^Q for W^I. Let π_Q denote the projection $G/B \to G/Q$ and recall that $W^Q = \{w \in W \mid \pi_Q^{-1}(X_{wQ}) = X_{wB}\}$.

Further, for $y \leq w$ in W, let $\mathcal{N}_{yQ,wQ} = (y(U_Q^-) \cap U^-)e_{yQ} \cap X_{wQ}$. This is a closed, T-stable, subvariety of $yU_Q^- e_Q \cap X_{wQ}$ and, similarly to [7, Lemma A4.(b)], one obtains a T-equivariant isomorphism

$$yU_Q^- e_Q \cap X_{wQ} \cong C_{yQ} \times \mathcal{N}_{yQ,wQ}.$$

Thus, we may call $\mathcal{N}_{yQ,wQ}$ a transversal to C_{yQ} in X_{wQ}.

1.3

Let $\mathcal{X} = \mathcal{X}(T)$ be the character group of T, let $\{\alpha^\vee,\ \alpha \in R\}$ be the set of coroots, and let $\mathcal{X}^+ = \{\lambda \in \mathcal{X} \mid (\lambda, \alpha^\vee) \geq 0,\ \ \forall\, \alpha \in \Delta\}$. For $\lambda \in \mathcal{X}$, let $\mathcal{L}(\lambda)$ denote the corresponding G-equivariant line bundle on G/B, and, for $\lambda \in \mathcal{X}^+$, let $V(\lambda) = H^0(G/B, \mathcal{L}(-\lambda))^*$ be the Weyl module with highest weight λ. It is generated by a B-stable line of weight λ, and its T-character is given by Weyl's character formula (see, for example, [6, II.2.13, II.5.11]). Similarly, if I is a subset of Δ, let $\mathcal{X}_I^+ = \{\lambda \in \mathcal{X} \mid (\lambda, \alpha^\vee) \geq 0,\ \ \forall\, \alpha \in I\}$ and, for $\lambda \in \mathcal{X}_I^+$, let $V_I(\lambda) = H^0(P_I/B, \mathcal{L}(-\lambda))^*$. This is the Weyl module for L_I with highest weight λ.

1.4

For future reference, let us record the following lemma.

Lemma. *Let Q be a parabolic subgroup containing B. Let $y \leq w$ in W^Q. Then π_Q induces an isomorphism $\mathcal{N}_{yB,wB} \cong \mathcal{N}_{yQ,wQ}$.*

Proof. Since $y \in W^Q$, then $R^+ \cap y(R^+) = R^+ \cap y(R^+ \setminus R_Q^+)$. This implies that $y(U^-) \cap U^- = y(U_Q^-) \cap U^-$. Let H denote this group. By the Bruhat decomposition, one has $\mathrm{Stab}_H(e_{yB}) = \{1\} = \mathrm{Stab}_H(e_{yQ})$ and hence π_Q induces a TH-equivariant isomorphism from He_{yB} onto He_{yQ}. Then, since $\pi_Q^{-1}(X_{wQ}) = X_{wB}$, one deduces that π_Q induces a T-equivariant isomorphism from $\mathcal{N}_{yB,wB} = He_{yB} \cap X_{wB}$ onto $He_{yQ} \cap X_{wQ} = \mathcal{N}_{yQ,wQ}$. The lemma is proved.

1.5

Let ℓ be a prime number different from $\mathrm{char}(k)$. For an algebraic variety X, let $\mathcal{IC}^\bullet(X)$ denote the middle intersection cohomology complex on X with coefficients in $\overline{\mathbb{Q}}_\ell$ and, for $i \in \mathbb{Z}$, let $\mathcal{IH}^i(X)$ denote the i-th cohomology sheaf of $\mathcal{IC}^\bullet(X)$ [3, Sect. 6], see also [8, Sect. 3]. We follow the normalisation of $\mathcal{IC}^\bullet(X)$ given in [8, 3.1(a)], that is, the restriction of $\mathcal{IC}^\bullet(X)$ to the smooth part of X is the constant sheaf $\overline{\mathbb{Q}}_\ell$ in degree zero. (This differs from the normalisation in [3, Definition 6.1(a)] by a shift in degree). For $x \in X$, let $\mathcal{IH}^i_x(X)$ denote the stalk of $\mathcal{IH}^i(X)$ at x. Then, following [7, Appendix], coupled with [8, Sects. 3–4], let us say that X is rationally smooth if $\mathcal{IH}^i_x(X) = 0$, for every $x \in X$ and $i > 0$. Note that if X is smooth then it is rationally smooth.

Let q be an indeterminate. We shall need the following notation. For a polynomial $P = \sum_i a_i\, q^i$ and a positive rational number r, let $P^{\leq r} = \sum_{i \leq r} a_i\, q^i$.

For $y \leq w$ in W, let $P_{y,w}(q)$ be the corresponding Kazhdan-Lusztig polynomial [7]. By [8, Theorem 4.3] (when $\mathrm{char}(k) > 0$) and [15, Corollaire 2.10], one has $P_{y,w}(q) = \sum_i \dim \mathcal{IH}^{2i}_{e_{yB}}(X_{wB})\, q^i$. Suppose that $y < w$ and that $P_{z,w} = 1$, for $y < z \leq w$. Let us then recall the following description of $P_{y,w}$, given in [7, Appendix]. Suppose that $\mathrm{char}(k) = p > 0$. Everything in sight is defined over the prime field $\mathbb{F}_p$ and one deduces from [7] the following result.

Lemma. *Let $y < w$ in W and let $d = \ell(w) - \ell(y)$.*

(a) *There exists a polynomial $K_{y,w}$, of degree d, such that, for every $r \geq 1$, the number of $\mathbb{F}_{p^r}$-rational points of $\mathcal{N}_{yB,wB} \setminus \{e_{yB}\}$ equals $K_{y,w}(p^r)$.*

(b) *If $\mathcal{N}_{yB,wB} \setminus \{e_{yB}\}$ is rationally smooth, then $P_{y,w} = (-K_{y,w})^{\leq (d-1)/2}$.*

Proof. The first assertion is a consequence of [7, 2.5, A4] and, since $P_{y,w}$ has degree at most $(d-1)/2$, the second assertion follows from the equation preceding Equation (5) in [7, Appendix].

2 Closures of orbits of highest weight vectors as transversals

2.1

For future use, let us record here the following lemma. We relax, in this subsection, the notation of Sect. 1.

Lemma. *Let G be a connected reductive group over k; choose a maximal unipotent subgroup $U \subset G$ and a maximal torus T normalising U. Let H be a subgroup of G containing U, and denote by P the normaliser of H in G. Then P contains TU, and H contains the derived subgroup of P. Moreover, H is generated by $U(T \cap H)$ and by the $U_{-\alpha}$ ($\alpha \in \Delta$) which it contains.*

Proof. The first statement is due to F. Knop ([11, Satz 2.1]); we recall his proof for the convenience of the reader. By a theorem of Chevalley, there exists a G-module V and a vector $v \in V$ such that H is the isotropy subgroup of the line kv. Decomposing v in V^U, we can write $v = \sum v_i$ where the v_i are eigenvectors of B with pairwise distinct weights χ_i. Let Q denote the intersection of the stabilisers of the lines kv_i (it is a parabolic subgroup of G). Then χ_i extends uniquely to a character of Q, and one has $H = \bigcap_{i,j} \mathrm{Ker}(\chi_j^{-1}\chi_i)$. Therefore, one has $Q' \subseteq H \subseteq Q$, where Q' denotes the derived subgroup of Q. Since $Q = Q'T$, it follows that $H = Q'(T \cap H)$. This implies the second assertion. Moreover, Q normalises H and hence $Q \subseteq N_G(H) = P$. On the other hand, P normalises $R_u(H)$, the unipotent radical of H. But one has $R_u(H) = R_u(Q)$ and, since $N_G(R_u(Q)) = Q$, one deduces that $P \subseteq Q$. Thus, $P = Q$ and the first assertion follows.

2.2

Let the notation of Sect. 1 be in force again. In this paragraph, we recall some facts about orbit closures of a highest weight vector in a Weyl module. Let $\lambda \in X(T)^+$ and let P be the associated parabolic subgroup of G (*i.e.*, P is generated by B and the $U_{-\alpha}$, for those $\alpha \in \Delta$ such that $(\lambda, \alpha^\vee) = 0$). Then λ extends to a character of P, and the associated line bundle $\mathcal{L}_P(-\lambda)$ on G/P is very ample. The dual space of $H^0(G/P, \mathcal{L}_P(-\lambda))$ is the Weyl module $V(\lambda)$, and the affine cone over G/P embedded in $\mathbb{P}V(\lambda)$ is the orbit closure of a highest weight vector. Denote by $\mathcal{C}(\lambda)$ this affine cone; then $\mathcal{C}(\lambda)$ is normal by [14, Theorem 3].

Consider now $G \times^P k_\lambda$, the total space of the line bundle $\mathcal{L}_P(\lambda)$. Identify k_λ with the λ-weight space in $V(\lambda)$. Then we have a map

$$\phi : G \times^P k_\lambda \to \mathcal{C}(\lambda)$$

induced by $(g, v) \mapsto gv$. We claim that ϕ is proper and induces an isomorphism $G \times^P (k_\lambda \setminus \{0\}) \to \mathcal{C}(\lambda) \setminus \{0\}$ (in particular, ϕ is birational). Indeed, consider the total space $O_{\mathbb{P}V(\lambda)}(-1)$ of the tautological line bundle over $\mathbb{P}V(\lambda)$. The canonical map

$$\Phi : O_{\mathbb{P}V(\lambda)}(-1) \to V(\lambda)$$

is the blow-up of the origin in $V(\lambda)$. In particular, Φ is proper and its restriction to the complement of the zero section is an isomorphism on the complement of the origin. Moreover, for G/P embedded into $\mathbb{P}V(\lambda)$, the space $O_{G/P}(-1)$ is the total space of $\mathcal{L}_P(\lambda)$, that is, $G \times^P k_\lambda$, and ϕ is the restriction of Φ. This proves our claim.

Since $\mathcal{C}(\lambda)$ is normal, it follows from Zariski's main theorem that

$$k[\mathcal{C}(\lambda)] \cong k[G \times^P k_\lambda] = \bigoplus_{n \geq 0} V(n\lambda)^*. \tag{†}$$

For later use in Sect. 4, let us record here the following generalisation. Let $\lambda_1, \ldots, \lambda_r \in X(T)^+$, let $P_1, \ldots, P_r$ be the associated parabolic subgroups, and let $Q = P_1 \cap \cdots \cap P_r$. Let $V = \bigoplus_{i=1}^r V(\lambda_i)$, let E be the Q-submodule spanned by the highest weight vectors, and let $\mathcal{C}(\lambda_1, \ldots, \lambda_r) = GE$ (which is closed since G/Q is complete). Then the $\mathcal{L}_{P_i}(-\lambda_i)$ define a closed immersion of G/Q into $\mathbb{P}V(\lambda_1) \times \cdots \times \mathbb{P}V(\lambda_r)$, and the corresponding multicone identifies with $\mathcal{C}(\lambda_1, \ldots, \lambda_r)$. Also, $G \times^Q E$ is the total space of the vector bundle $\bigoplus_{i=1}^r \mathcal{L}_Q(\lambda_i)$. Further, along the same lines as above, one can show that the natural map $\phi : G \times^Q E \to V$, induced by $(g, v) \mapsto gv$, is proper and induces an isomorphism $G \times^Q E^\times \xrightarrow{\cong} GE^\times$, where $E^\times$ denotes the Q-stable, open subvariety of E consisting of those vectors whose projection onto $V(\lambda_i)$ is non-zero, for every $i = 1, \ldots, r$. (See also the proof of [9, Theorem 1] for a more general statement).

Moreover, by [9, Theorem 2], $\mathcal{C}(\lambda_1, \ldots, \lambda_r)$ is normal. Thus, by Zariski's main theorem, it follows that

$$k[\mathcal{C}(\lambda_1, \ldots, \lambda_r)] \cong k[G \times^Q E] = \bigoplus_{n_1,\ldots,n_r \geq 0} V(n_1\lambda_1 + \cdots + n_r\lambda_r)^*. \quad (\dagger\dagger)$$

2.3

We can now prove the following

Proposition. *Let $I \subset \Delta$ and let $P = P_I$, $L = L_I$. Let $\beta \in \Delta \setminus I$. Then $U_P^- e_P \cap \overline{Pe_{s_\beta P}}$, which is an L-stable open neighbourhood of e_P in $\overline{Pe_{s_\beta P}}$, is L-isomorphic to $\mathcal{C}_I(-\beta)$, the orbit closure of a highest weight vector in the Weyl module $V_I(-\beta)$.*

Proof. Let $Y = U_P^- e_P \cap \overline{Pe_{s_\beta P}}$. Observe that Y is normal: indeed, it is open in the Schubert variety $\overline{Pe_{s_\beta P}}$, and the latter is normal by [14, Theorem 3]. Let P_0 be the parabolic subgroup of L associated with the dominant weight $-\beta$. We will construct a proper birational morphism $\phi : L \times^{P_0} k_{-\beta} \to Y$. By Zariski's main theorem, it follows that $k[Y] \cong k[L \times^{P_0} k_{-\beta}]$. But the latter is isomorphic to $k[\mathcal{C}_I(-\beta)]$ by (†) applied to L. Because both Y and $\mathcal{C}_I(-\beta)$ are affine, we conclude that $Y \cong \mathcal{C}_I(-\beta)$.

Choose $u \in U_\beta^\times$ and set $x = ue_{s_\beta P}$. Note first that $Tx = U_\beta^\times e_{s_\beta P} = U_{-\beta}^\times e_P$. Hence $e_{s_\beta P}$ and e_P belong to $\overline{Tx}$ (the closure of Tx in G/Q). Let $U_{(\beta)}$ denote the unipotent radical of the minimal parabolic subgroup P_β. Note also that $e_{s_\beta P}$ is fixed by $U_{(\beta)}$ and hence, since $P = L\, U_\beta\, U_{(\beta)}$, one has $Pe_{s_\beta P} = L\, U_\beta e_{s_\beta P} = Lx \cup Le_{s_\beta P}$. It follows that $\overline{Pe_{s_\beta P}} = \overline{Lx}$. Thus, $Y = \overline{Lx} \cap U_P^- e_P$.

Let L_x (resp. T_x) denote the stabiliser of x in L (resp. T). For any $\alpha \in R_I^+$, one has $u^{-1} U_\alpha u \subseteq U_{(\beta)}$ and hence U_α stabilises x. Therefore, L_x contains $U \cap L$ and, by Lemma 2.1, it follows that L_x is generated by $(L \cap U)T_x$ together with the $U_{-\gamma}$ ($\gamma \in I$) which it contains. But for $\gamma \in I$, one has $U_{-\gamma} x = U_{-\gamma} u e_{s_\beta P} = u U_{-\gamma} e_{s_\beta P} = u s_\beta U_{-s_\beta \gamma} e_P$. Since $\beta \notin I$ then $s_\beta \gamma \in R^+$ and hence one deduces that

$$U_{-\gamma} \subseteq L_x \iff U_{-s_\beta \gamma} \subseteq P \iff -s_\beta \gamma \in -R_I^+ \iff (\gamma, \beta^\vee) = 0.$$

Let $k_{-\beta}$ denote the one-dimensional representation of P_0 associated with the character $-\beta$. It follows from the above discussion that L_x is the kernel of this representation. This implies, in particular, that $P_0 x = Tx = U_{-\beta}^\times e_P$. Since $U_{-\beta}\, e_P$ is a closed subset of $U_P^- e_P$, one deduces that $\overline{P_0 x} \cap U_P^- e_P =$

$U_{-\beta}\, e_P = Tx \sqcup \{e_P\}$ and, since L/P_0 is complete, it follows that Y equals $L(\overline{P_0 x} \cap U_P^- e_P) = Lx \sqcup \{e_P\}$.

Choose an isomorphism of algebraic groups $\theta_{-\beta} : k \xrightarrow{\cong} U_{-\beta}$, such that $x = \theta_{-\beta}(1)$. Consider the L-equivariant morphism $\phi : L \times^{P_0} k_{-\beta} \longrightarrow Y$, $(g, z) \mapsto g\theta_{-\beta}(z)e_P$. Then, clearly, ϕ is well-defined and, since L/P_0 is complete, ϕ is proper. Finally, let us prove that ϕ is birational. First, it is easily seen that the morphism $L \to L \times^{P_0} k_{-\beta}$, induced by $g \mapsto (g, 1)$, induces an isomorphism $\pi : L/L_x \xrightarrow{\cong} L \times^{P_0} (k_{-\beta} \setminus \{0\})$, and that $\phi \circ \pi$ is the natural map $L \to Lx$. Further, the latter is separable, because $k(Lx) = k(Px) = k(Pe_{s_\beta P})$ and, by the Bruhat decomposition, the extension $k(Pe_{s_\beta P}) \subset k(P)$ is separable; but $k(P)$ contains $k(L)$. This proves that ϕ is birational.

2.4

Keep notation as in 2.3 and let $d = \dim \overline{Pe_{s_\beta P}}$, and $I_0 = \{\alpha \in I \mid (\alpha, \beta^\vee) = 0\}$. By Proposition 2.3, one has $d = 1 + \dim L/P_0 = 1 + \#(R_I^+ \setminus R_{I_0}^+)$. Note that if $d = 1$ then $\overline{Pe_{s_\beta P}} \cong \mathbb{P}^1$. So, suppose that $d > 1$. For any subset A of W, let $H(A, q) = \sum_{w \in A} q^{\ell(w)}$. As usual, set $\rho = (1/2) \sum_{\alpha \in R^+} \alpha$. Then, one obtains the following corollary.

Corollary. (a) *The tangent space $T_{e_P}(\overline{Pe_{s_\beta P}})$ is L-isomorphic to $V_I(-\beta)$.*

(b) *The multiplicity of $\overline{Pe_{s_\beta P}}$ at e_P equals* $(d-1)! \displaystyle\prod_{\gamma \in R_I^+ \setminus R_{I_0}^+} \frac{(-\beta, \gamma^\vee)}{(\rho, \gamma^\vee)}$.

(c) $\overline{Pe_{s_\beta P}}$ *is smooth if and only if β is adjacent to a unique connected component J of I, J is of type A_{d-1} or $C_{d/2}$ (if d is even), and $J \sqcup \{\beta\}$ has no branch point and has β as a short extremity.*

(d) *One has* $$P_{w_I, w_I w_{I_0} s_\beta w_I} = \left((1-q) \frac{H(W_I, q)}{H(W_{I_0}, q)} \right)^{\leq (d-1)/2}.$$

Proof. Let $V = V_I(-\beta)$ and let v be a highest weight vector in V. By Proposition 2.3, $T_{e_P}(\overline{Pe_{s_\beta P}})$ is isomorphic, as an L-module, to $T_0(\overline{Lv})$. But $T_0(\overline{Lv})$ is an L-stable subspace of V containing v, and moreover v generates V as an L-module. It follows that $T_0(\overline{Lv}) = V_I(-\beta)$. This proves assertion (a).

Let $Y = U_P^- e_P \cap \overline{Pe_{s_\beta P}}$ and let $\mathfrak{m}$ denote the maximal ideal of $k[Y]$ corresponding to e_P. Then $k[Y] \cong \bigoplus_{n \geq 0} V_I(-n\beta)^*$, by Proposition 2.3, together with 2.2(†) applied to L, and under this isomorphism one has $\mathfrak{m} \cong \bigoplus_{n \geq 1} V_I(-n\beta)^*$. Further, by [14, Theorem 1.ii)], the multiplication map

$$V_I(-\beta)^* \otimes V_I(-n\beta)^* \to V_I(-(n+1)\beta)^*$$

is surjective, for $n \geq 0$, and this implies that $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong V_I(-n\beta)^*$, for every $n \geq 1$. Thus, by Weyl's dimension formula, one obtains that

$$\begin{aligned}\dim\bigl(\mathfrak{m}^n/\mathfrak{m}^{n+1}\bigr) &= \prod_{\gamma\in R_I^+} \frac{(-n\beta+\rho,\gamma^\vee)}{(\rho,\gamma^\vee)} \\ &= n^{d-1} \prod_{\gamma\in R_I^+\setminus R_{I_0}^+} \frac{-(\beta,\gamma^\vee)}{(\rho,\gamma^\vee)} + O(n^{d-2}),\end{aligned}$$

and assertion (b) follows.

Let us prove assertion (c). First, $\overline{Pe_{s_\beta P}}$ is smooth if and only if it is smooth at e_P and, by Proposition 2.3, this is the case if and only if $\overline{Lv}$ is smooth at 0. But we just saw that $T_0(\overline{Lv}) = V$ and, since $\overline{Lv} = Lv \cup \{0\}$, it follows that $\overline{Lv}$ is smooth at 0 if and only if $Lv = V \setminus \{0\}$.

Let J denote the union of the connected components of I to which β is adjacent and let π denote the representation of L on V. Then, clearly, π maps the derived subgroup L'_J onto the derived subgroup of $\pi(L)$, and the restriction of π to L'_J has a finite kernel. Note that, if J' is a connected component of J then $J' \sqcup \{\beta\}$ is connected and hence, in particular, J' is not of type G_2. Thus, it follows from (the proof of) [10, Satz 1] that $Lv = V \setminus \{0\}$ if and only if J is connected and of type A_{d-1} or $C_{d/2}$ (if d is even), and the restriction of $-\beta$ to $T \cap H$ is a fundamental weight corresponding to a short extremity of J. This proves assertion (c).

Now, to assertion (d). Using the Bruhat decomposition, one first obtains that $\pi_P^{-1}(e_P) = P/B = X_{w_I B}$ and $\pi_P^{-1}(X_{s_\beta P}) = \overline{Bs_\beta P}/B = X_{s_\beta w_I B}$. Let $w = w_I w_{I_0} s_\beta w_I$. We claim that $\pi_P^{-1}(\overline{Ps_\beta P}/P) = X_{wB}$. Since the former equals $PX_{s_\beta w_I B}$, and since $w_I w_{I_0} \in W_I$, it suffices to prove that X_{wB} is P-stable. Thus, it suffices to prove that $w^{-1}\alpha \in R^-$, for every $\alpha \in I$. This is easily checked, and the claim follows.

Thus, by Lemma 1.4, $\mathcal{N}_{w_I B, wB} \cong \mathcal{N}_{P, wP}$, and, by Proposition 2.3, the latter is smooth outside e_P. Thus, we may apply the argument of 1.5 to compute $P_{w_I, w}$. So, suppose that $\mathrm{char}(k) = p > 0$. By Proposition 2.3, $\mathcal{N}_{P,wP} \setminus \{e_P\}$ is a $k^\times$-fibration over the flag variety L/P_0 and hence, using the Bruhat decomposition of L/P_0, one deduces that, for every $r \geq 1$,

$$(p^r-1)\,\#\{\mathbb{F}_{p^r}\text{-rational points of } L/P_0\} = (p^r-1)\frac{H(W_I,p^r)}{H(W_{I_0},p^r)}.$$

By Lemma 1.5(b), this implies assertion (d).

Remark. The most effective way to compute $P_{w_I,\, w_I w_{I_0} s_\beta w_I}$ explicitly is as follows. Let $n = |I|$ (resp. $n_0 = |I_0|$) and let $a_1, \ldots, a_n$ (resp. $b_1, \ldots, b_{n_0}$) be the exponents of W_I (resp. W_{I_0}). It is well-known that $H(W_I, q) =$

$(1-t)^{-n}\prod_{i=1}^{n}(1-t^{a_i})$ (see, for example, [5, Theorem 3.15]) and one has an analogous formula for $H(W_{I_0},q)$. Thus, one obtains

$$P_{w_I,\, w_I w_{I_0} s_\beta w_I} = \left((1-t)^{1+n_0-n} \frac{\prod_{i=1}^{n}(1-t^{a_i})}{\prod_{i=1}^{n_0}(1-t^{b_i})} \right)^{\leq (d-1)/2}.$$

2.5

Before we prove the main result of this section, we need the following lemma. Let Q be a parabolic subgroup of G containing B and let $y \leq w$ in W^Q. Let $C_{[yQ,wQ]}$ denote the union of the B-orbits C_{zQ}, for $z \in [y,w]$. This is a B-stable open subset of X_{wQ} containing C_{yQ} as unique closed B-orbit.

Lemma. *$y(U_Q^-)e_{yQ} \cap X_{wQ}$ is the unique T-stable, open affine subset of X_{wQ} containing e_{yQ}.*

Proof. Let $\Omega = y(U_Q^-)e_{yQ} \cap X_{wQ}$ and let Ω' be a second T-stable, open affine subset of X_{wQ} containing e_{yQ}. Then $Z := \Omega \setminus \Omega'$ is a closed, T-stable, subset of Ω which does not contain e_{yQ}. Since e_{yQ} is the unique closed T-orbit in $y(U_Q^-)e_{yQ}$, it follows that $Z = \emptyset$ and hence $\Omega \subseteq \Omega'$.

Therefore, the algebra of T-invariant regular functions $k[\Omega']^T$ injects into $k[\Omega]^T$. But the latter equals k, because e_{yQ} is the unique closed T-orbit in Ω. So $k[\Omega']^T = k$ and hence Ω' contains a unique closed T-orbit, which must be the fixed point e_{yQ}. Now the same argument as above gives $\Omega' \subseteq \Omega$. This proves the lemma.

2.6

Let Q be a parabolic subgroup of G containing B. First, we observe that, for any $z \in W$, the stabiliser in G of C_{zQ} (resp. of X_{zQ}) is the parabolic subgroup generated by B and the s_α, for $\alpha \in \Delta \cap y(R_Q)$ (resp. by B and the s_α, for $\alpha \in \Delta \cap y(R_Q \cup R^-)$). This fact, which follows easily from the Bruhat decomposition, will be used repeatedly in the sequel.

Now, let $y \leq w$ in W^Q. Let I be a subset of $\Delta \cap y(R_Q)$ and let $P = P_I$, $L = L_I$. Then P is contained in the stabiliser of C_{yQ}, and L is contained in $P \cap y(Q) := P_{yQ}$, the stabiliser in P of the point e_{yQ}. Also, one deduces from the Bruhat decomposition that $P/P_{yQ} \cong C_{yQ}$.

Further, let us suppose that :

$$X_{wQ} = P X_{s_\beta yQ}, \quad \text{for some } \beta \in \Delta \cap y(R^+ \setminus R_Q^+).$$

Let $\mathcal{C}_I(-\beta)$ denote the orbit closure of a highest weight vector in $V_I(-\beta)$.

Theorem. (a) *The morphism* $\varphi : \overline{Ps_\beta P}/P_{yQ} \longrightarrow G/Q$*,* $gP_{yQ} \mapsto ge_{yQ}$ *induces a* P*-equivariant isomorphism from* $\overline{Ps_\beta P}/P_{yQ}$ *onto* $C_{[yQ,wQ]}$ *and hence one has a locally trivial fibration* $\pi : C_{[yQ,wQ]} \longrightarrow \overline{Pe_{s_\beta}P}/P$*, with fiber* $P/P_{yQ} \cong C_{yQ}$*.*

(b) *One has* L*-equivariant isomorphisms :* $y(U_Q^-)e_{yQ} \cap X_{wQ} \cong \mathcal{C}_I(-\beta) \times C_{yQ}$ *and, more precisely,* $\mathcal{N}_{yQ,wQ} \cong \mathcal{C}_I(-\beta)$*.*

Proof. Clearly, φ is a P-equivariant morphism; let us describe its image $\operatorname{Im}\varphi$. Since $\overline{Ps_\beta P} = P \cup Ps_\beta P$, one has $\operatorname{Im}\varphi = Pe_{yQ} \cup Ps_\beta Pe_{yQ} = C_{yQ} \cup Ps_\beta C_{yQ}$. Moreover, since $s_\beta y > y$, one has, by the Bruhat decomposition, $Bs_\beta\, C_{yQ} = C_{s_\beta yQ}$. It follows that $\operatorname{Im}\varphi = C_{yQ} \sqcup PC_{s_\beta yQ}$. Observe that $C_{s_\beta yQ}$ is contained in $C_{[yQ,wQ]}$ and that the latter is P-stable (because X_{wQ} and C_{yQ} are P-stable). Therefore, one has $\operatorname{Im}\varphi \subseteq C_{[yQ,wQ]}$. Observe also that $\varphi^{-1}(u)$ is a single point for all $u \in C_{yQ}$.

Let us prove that φ is proper. Define morphisms

$$\overline{Ps_\beta P}/P_{yQ} \xrightarrow{\varphi_1} (\overline{Ps_\beta P}/P) \times C_{[yQ,wQ]} \xrightarrow{\varphi_2} C_{[yQ,wQ]}$$

by $\varphi_1(gP_{yQ}) = (gP, ge_{yQ})$ and $\varphi_2((gP, ge_{yQ})) = ge_{yQ}$. Then $\varphi = \varphi_2 \circ \varphi_1$ and hence, since $\overline{Ps_\beta P}/P$ is complete, φ_2 is proper. Further, φ_1 is injective. For, if $(gP, ge_{yQ}) = (g'P, g'e_{yQ})$ then $g^{-1}g' \in P_{yQ}$. Finally, one has $\operatorname{Im}\varphi_1 = \{(gP, u) \mid u \in gC_{yQ}\}$ and, since C_{yQ} is closed in $C_{[yQ,wQ]}$, it follows that $\operatorname{Im}\varphi_1$ is closed. Thus, being injective with closed image, φ_1 is proper and the same is true for φ.

Let Z be the set of those $z \in \overline{Ps_\beta P}/P_{yQ}$ such that the fibre $\varphi^{-1}(\varphi(z))$ contains an infinite irreducible component passing through z. It is P-stable, since φ is P-equivariant. Further, by [4, Ex. II.3.22], Z is a closed subset of $\overline{Ps_\beta P}/P_{yQ}$ and hence, φ being proper, $\varphi(Z)$ is a closed, P-stable, subset of $C_{[yQ,wQ]}$. Note that $\varphi(Z) = \{u \in C_{[yQ,wQ]} \mid \varphi^{-1}(u) \text{ is infinite}\}$. On the other hand, we observed previously that $\varphi^{-1}(u)$ is a single point for all $u \in C_{yQ}$. Since C_{yQ} is the unique closed P-orbit in $\operatorname{Im}\varphi$, one deduces that $\varphi(Z) = \emptyset$. Thus, φ is quasi-finite. It follows, in particular, that $\dim \overline{Ps_\beta P}/P_{yQ} = \dim C_{[yQ,wQ]} = \dim C_{wQ}$ and hence the B-stable open subset $\varphi^{-1}(C_{wQ})$ is the disjoint union of B-orbits of dimension $\dim C_{wQ}$. Since $\overline{Ps_\beta P}/P_{yQ}$ is irreducible, it follows that $\varphi^{-1}(C_{wQ})$ is in fact a single B-orbit, namely the B-orbit of the point $x := w_I s_\beta P_{yQ}/P_{yQ}$.

Thus, φ induces a quasi-finite, B-equivariant, morphism from the open orbit Bx onto its image Be_{wQ}. This implies that B_x, the stabiliser in B of x, is a subgroup of finite index in B_{wQ}, the stabiliser in B of e_{wQ}. But B_{wQ} is connected, because it contains T, and it follows that $B_x = B_{wQ}$. Since, moreover, the orbit map $B \to C_{wQ}$ is separable, one deduces that φ induces an isomorphism $Bx \cong C_{wQ}$. Thus φ is birational.

Finally, by [14, Theorem 3], $C_{[yQ,wQ]}$ is a normal variety and hence, φ being proper, birational, and quasi-finite, it follows from Zariski's main theorem that φ is an isomorphism. This proves the first part of assertion (a), and the second part follows easily.

Let us prove assertion (b). Let $\Omega = y(U_Q^-)e_{yQ} \cap X_{wQ}$, let $\Omega' = \pi^{-1}(U_P^- e_P \cap \overline{Pe_{s_\beta P}})$, and let $\mathcal{U} = \{u \in U_P^- \mid ue_P \in \overline{Pe_{s_\beta P}}\}$. Then $\mathcal{U}$ identifies, via the map $u \mapsto ue_P$, with $U_P^- e_P \cap \overline{Pe_{s_\beta P}}$ and hence, by Proposition 2.3, one has $\mathcal{U} \cong \mathcal{C}_I(-\beta)$. Further, since π_P trivialises over the open affine subset $U_P^- e_P$ of G/P, one deduces that the map $(u,x) \mapsto ux$ induces an L-equivariant isomorphism $\phi : \mathcal{U} \times C_{yQ} \cong \Omega'$. Therefore, Ω' is an L-stable, open affine subset of $C_{[yQ,wQ]}$ containing e_{yQ} and hence, by Lemma 2.5, one has $\Omega' = \Omega$. Therefore, one has an isomorphism $\phi : \mathcal{U} \times C_{yQ} \to \Omega$, $(u,x) \to ux$, with $\mathcal{U} \cong \mathcal{C}_I(-\beta)$. This proves the first isomorphism.

For the second one, observe that $\mathcal{U} \subseteq U_P^- \cap P_{I'}$, where $I' = I \cup \{\beta\}$. Since $y^{-1}(R_I^-) \subseteq R_Q$ and $y^{-1}(-\beta) \in R^- \setminus R_Q^-$, then $y^{-1}(R_{I'}^- \setminus R_I^-) \subseteq R^- \setminus R_Q^-$ and hence $(U_P^- \cap P_{I'})e_{yQ} \subseteq (y(U_Q^-) \cap U^-)e_{yQ}$. One deduces that ϕ maps isomorphically $\mathcal{U} \times \{e_{yQ}\}$ onto a closed subset of $\mathcal{N}_{yQ,wQ}$. Since, by assertion (a), they have the same dimension, it follows that $\phi(\mathcal{U} \times \{e_{yQ}\}) = \mathcal{N}_{yQ,wQ}$. This completes the proof of the theorem.

2.7

Keep the notation of 2.6. Let $N_{yQ,wQ} = T_{yQ}\mathcal{N}_{yQ,wQ}$; it is an L-submodule of $T_{yQ}(G/Q)$, isomorphic to the normal space to C_{yQ} in X_{wQ} at e_{yQ}. Let $I_0 = \{\alpha \in I \mid (\alpha, \beta^\vee) = 0\}$ and let $d = \ell(w) - \ell(y) = \dim X_{wQ} - \dim X_{yQ} = 1 + \#(R_I^+ \setminus R_{I_0}^+)$. Let $\mathrm{mult}_{yQ} X_{wQ}$ denote the multiplicity of X_{wQ} at e_{yQ}. Let us then derive the following corollary.

Corollary. (a) $N_{yQ,wQ} \cong V_I(-\beta)$.

(b) $\mathrm{mult}_{yQ} X_{wQ} = (d-1)! \displaystyle\prod_{\gamma \in R_I^+ \setminus R_{I_0}^+} \frac{(-\beta, \gamma^\vee)}{(\rho, \gamma^\vee)}$.

(c) $P_{y,w}(q) = \left((1-q) \dfrac{H(W_I, q)}{H(W_{I_0}, q)} \right)^{\leq (d-1)/2}$.

(d) *One has* $w = w_I w_{I_0} s_\beta y$.

Proof. Assertions (a) and (b) follow immediately from the theorem, combined with Corollary 2.4. Let us prove assertion (c). Since $y, w \in W^Q$, it follows from Lemma 1.4, coupled with Theorem 2.6.(b), that $\mathcal{N}_{yB,wB} \cong \mathcal{N}_{yQ,wQ} \cong \mathcal{C}_I(-\beta)$. Thus, $\mathcal{N}_{yB,wB} \setminus \{e_{yB}\}$ is smooth and hence we can apply Lemma 1.5. So, we may assume that $\mathrm{char}(k) = p > 0$. But then, for

$r \geq 1$, the number of $\mathbb{F}_{p^r}$-rational points of $\mathcal{C}_I(-\beta)$ was computed in the proof of Corollary 2.4 and hence assertion (c) follows.

Finally, let us prove assertion (d). Let $z = w_I w_{I_0} s_\beta y = w_I s_\beta w_{I_0} y$. Then, since L fixes ye_Q, one has $ze_Q = w_I s_\beta y e_Q = we_Q$ and hence $z \in wW_Q$. Since $w \in W^Q$, by assumption, the equality $z = w$ will follow if we prove that $w\alpha \in R^-$, for all $\alpha \in R_Q^+$. Recall that, by hypothesis, $y^{-1}\beta \in R^+ \setminus R_Q^+$. Suppose, for a contradiction, that $w\alpha \in R^+$, for some $\alpha \in R_Q^+$. Since $y\alpha \in R^- \setminus \{-\beta\}$, then $s_\beta y\alpha \in R^-$ and hence the assumption $w\alpha \in R^+$ implies that $s_\beta y \alpha \in R_I^- \setminus R_{I_0}^-$. It follows that $(y\alpha, \beta^\vee) < 0$ and, since $R_I^- \subseteq yR_Q^+$, one obtains that $\beta = (y\alpha, \beta^\vee)^{-1}\,(y\alpha - s_\beta y\alpha)$ belongs to $\mathbb{Q}(yR_Q) \cap R = yR_Q$. This is a contradiction and the proof of the corollary is complete.

3 Application to the minuscule case

3.1

Throughout this section, we suppose that G is quasi-simple and that Q is the maximal parabolic subgroup associated with ω, a minuscule fundamental weight. We shall also assume that G is simply-laced, which entails no loss of generality. For, if G is of type B_n or C_n and if P is the maximal parabolic subgroup corresponding to the unique minuscule fundamental weight, it is well-known that $X := G/P$ identifies with G'/P', where G' is of type D_{n+1} or A_{2n-1}, respectively, and P' is a maximal parabolic corresponding to a minuscule fundamental weight. Moreover, let B' be a Borel subgroup in P' and let $B = G \cap B'$. By the Bruhat decomposition, B' and B have the same number of orbits in X and it follows that the orbits are the same under B' or B. Thus, the Schubert varieties are the same in G/P and in G'/P'.

Under the above assumptions, we shall prove that, for $y \leq w$, the hypotheses of 2.6 are always satisfied if X_{yQ} is an irreducible component of the singular locus of X_{wQ}. Thus, our previous results will give a description of the singularity of X_{wQ} along X_{yQ}. The starting point of the proof is the fact that, Q being minuscule, the Bruhat order on W^Q is generated by simple reflections [12, Lemma 1.14].

3.2

For $w \in W^Q$, let $\mathrm{Bd}(X_{wQ})$ denote the boundary of X_{wQ}, that is, $\mathrm{Bd}(X_{wQ}) = X_{wQ} \setminus P_J e_{wQ}$, where P_J denotes the stabiliser of X_{wQ}. Also, let us introduce the usual partial order on $\mathcal{X}(T)$, defined by: $\mu \leq \lambda \iff \lambda - \mu \in \mathbb{N}R^+$.

Lemma. *Let $y \leq w$ in W^Q.*

(a) *Suppose that X_{yQ} is an irreducible component of* $\mathrm{Bd}(X_{wQ})$. *Then there exists a unique simple root β such that $X_{yQ} \subset X_{s_\beta yQ} \subseteq X_{wQ}$ and one has $X_{wQ} = PX_{s_\beta yQ}$, where $P = \mathrm{Stab}(X_{wQ}) \cap \mathrm{Stab}(C_{yQ})$.*

(b) *The irreducible components of* $\mathrm{Bd}(X_{wQ})$ *are exactly the $X_{s_\gamma wQ}$, for γ a minimal element of the set $\{\alpha \in R^+ \mid X_{s_\alpha wQ} \subseteq \mathrm{Bd}(X_{wQ})\}$.*

Proof. Let $J = \Delta \cap w(R^- \cup R_Q)$. Then $\mathrm{Stab}(X_{wQ}) = P_J$. Let X_{yQ} be an irreducible component of $\mathrm{Bd}(X_{wQ})$. Observe that X_{yQ} is P_J-stable. By [12, Lemma 1.14], there exists $\beta \in \Delta$ such that $X_{yQ} \subset X_{s_\beta yQ} \subseteq X_{wQ}$. Note, in particular, that $\beta \notin J$. Let $I = J \cap y(R_Q)$ and let $z = w_I s_\beta y$. Note that $e_{zQ} \in P_I e_{s_\beta yQ} \subseteq X_{wQ}$. Let us prove that X_{zQ} is P_J-stable. By 1.2, it suffices to prove that $(z\omega, \alpha^\vee) \leq 0$, for $\alpha \in J$. Observe that $s_\beta y\omega = y\omega - \beta$ and, since $w_I y\omega = y\omega$, it follows that $(z\omega, \alpha^\vee) = (y\omega, \alpha^\vee) - (w_I\beta, \alpha^\vee)$.

Also, since X_{yQ} is P_J-stable, then $(y\omega, \alpha^\vee) \leq 0$, for $\alpha \in J$. If $\alpha \in J \setminus I$ then $(y\omega, \alpha^\vee) = -1$. Moreover, since G is simply-laced and $w_I\beta \neq \alpha$, one has $-(w_I\beta, \alpha^\vee) \leq 1$. So one obtains in this case $(z\omega, \alpha^\vee) \leq 0$. On the other hand, if $\alpha \in I$ then $(y\omega, \alpha^\vee) = 0$ and $-w_I\alpha \in I$ and hence, since $\beta \notin J$, one also obtains $(z\omega, \alpha^\vee) \leq 0$. This proves that X_{zQ} is P_J-stable and it follows that $X_{zQ} = X_{wQ}$.

Thus, one obtains that $w\omega = y\omega - w_I\beta = s_{w_I\beta} y\omega$, and this implies that $\beta = w_I(y\omega - w\omega)$ is uniquely determined by w and y. This proves assertion (a). Further, setting $\gamma = w_I\beta$, one has $\gamma \in R^+$ and $y\omega = w\omega + \gamma = s_\gamma w\omega$.

Now, let $\delta \in R^+$. Suppose that $X_{s_\delta wQ} \subseteq \mathrm{Bd}(X_{wQ})$. First, this implies that $(w\omega, \delta^\vee) < 0$ and hence, since ω is minuscule, that $s_\delta w\omega = w\omega + \delta$. Then, one deduces from [12, Lemma 1.18] that

$$X_{s_\delta wQ} \subseteq X_{s_\gamma wQ} \iff s_\gamma w\omega \leq s_\delta w\omega \iff \gamma \leq \delta.$$

This completes the proof of the lemma.

3.3

Combining the previous lemma with the results of Sect. 2, we obtain the following proposition. For a rational number r, let $[r]$ denotes the largest integer not greater than r.

Proposition. *Let $y, w \in W^Q$ and let $J = \Delta \cap w(R^- \cup R_Q)$.*

(a) $\mathrm{Bd}(X_{wQ})$ *equals the singular locus of X_{wQ}.*

(b) *Suppose that X_{yQ} is an irreducible component of* $\mathrm{Bd}(X_{wQ})$. *Let β be the unique simple root such that $X_{yQ} \subset X_{s_\beta yQ} \subseteq X_{wQ}$ and let I be the union of the connected components of $J \cap y(R_Q)$ to which β is adjacent.*

Then the normal space $N_{yQ,wQ}$ is isomorphic to the L_I-module $V_I(-\beta)$, and $\mathcal{N}_{yQ,wQ}$ identifies with the closure of the L_I-orbit of a highest weight vector in this module.

(c) *Thus, $\mathcal{N}_{yQ,wQ}$ is determined by the pair (I, I'), where $I' = I \sqcup \{\beta\}$, and, therefore, the only possibilities are the following.*

Case 1). *I is of type $A_p \times A_q$ and I' of type A_{p+q+1}. Then $\mathcal{N}_{yQ,wQ}$ is isomorphic to the cone of decomposable tensors in $k^{p+1} \otimes k^{q+1}$ and has dimension $p + q + 1$. One has*

$$\mathrm{mult}_{yQ} X_{wQ} = \binom{p+q}{p}, \qquad P_{y,w} = \sum_{i=0}^{\mathrm{Min}(p,q)} t^i.$$

Case 2). *I is of type A_n and I' of type D_{n+1}. Then $\mathcal{N}_{yQ,wQ}$ is isomorphic to the cone of decomposable vectors in $\Lambda^2 k^{n+1}$ and has dimension $2n - 1$. One has*

$$\mathrm{mult}_{yQ} X_{wQ} = \frac{1}{n}\binom{2n-2}{n-1}, \qquad P_{y,w} = \sum_{i=0}^{[\frac{n-1}{2}]} t^{2i}.$$

Case 3). *I is of type D_n and I' of type D_{n+1}. Then $\mathcal{N}_{yQ,wQ}$ is isomorphic to a non-degenerate quadratic cone in k^{2n} and has dimension $2n - 1$. One has*

$$\mathrm{mult}_{yQ} X_{wQ} = 2, \qquad P_{y,w} = 1 + t^{n-1}.$$

Case 4). *I is of type D_5 and I' of type E_6. Then $N_{yQ,wQ}$ identifies with $V \cong k^{16}$, a half-spin representation of Spin(10), and $\mathcal{N}_{yQ,wQ}$ is isomorphic to the cone of pure half-spinors in V and has dimension 11. One has*

$$\mathrm{mult}_{yQ} X_{wQ} = 12, \qquad P_{y,w} = 1 + t^3.$$

Case 5). *I is of type E_6 and I' of type E_7. Then $N_{yQ,wQ}$ identifies with $V \cong k^{27}$, a minuscule representation of E_6, and $\mathcal{N}_{yQ,wQ}$ is isomorphic to the orbit closure of a highest weight vector in V and has dimension 17. One has*

$$\mathrm{mult}_{yQ} X_{wQ} = 78, \qquad P_{y,w} = 1 + t^4 + t^8.$$

Proof. Let X_{yQ} be an irreducible component of $\mathrm{Bd}(X_{wQ})$. Let β be the unique simple root such that $X_{yQ} \subset X_{s_\beta yQ} \subseteq X_{wQ}$, let I be the union of the connected components of $J \cap y(R_Q)$ to which β is adjacent, and let $I' = I \sqcup \{\beta\}$. Let us prove that e_{yQ} is a singular point of X_{wQ}. By (the proof of) Corollary 2.4.(c), it suffices to check that we are not in the situation where I is of type A_n and I' is of type A_{n+1}. Suppose, for a contradiction, that this is the case. Then

$$w_I\beta = \beta + \sum_{\alpha\in I} \alpha. \qquad (*)$$

On the other hand, the hypotheses imply that $(w\omega, \beta^\vee) = 1$ and $(w_I w\omega, \beta^\vee) = -1$. Thus, in particular, $w_I w\omega \neq w\omega$ and hence there exists $\alpha \in I$ such that $(w\omega, \alpha^\vee) = -1$. Moreover, since ω is minuscule and since I is connected, there exists only one such α (otherwise, there would exist $\gamma \in R_I^+$ such that $(w\omega, \gamma^\vee) \geq 2$) and hence $(*)$ implies that $(w\omega, w_I\beta^\vee) = 0$, which is a contradiction. This proves assertion (a). Assertion (b) then follows by combining Lemma 3.2.(a) and Theorem 2.6.

Let us prove assertion (c). First, since β is adjacent to every connected component of I, then I' is connected. Thus, since G is assumed to be simply-connected, I' is of type A, D, or E. Moreover, we claim that ω_β, considered as a fundamental weight of I', is minuscule. For, since $(y\omega, \beta^\vee) = 1$ and $(y\omega, \alpha^\vee) = 0$, for $\alpha \in I$, then $(y\omega, \gamma^\vee) = (\omega_\beta, \gamma^\vee)$, for all $\gamma \in R_{I'}$. The claim follows, since ω is minuscule. By inspection, one then obtains the possibilities 1)–5). Moreover, each possibility occurs by taking, for example, G of type I', $Q = P_I$ and $X_{wQ} = \overline{P_I e_{s_\beta Q}}$. Finally, all the statements and computations in cases 1)–5) are immediate consequences of Theorem 2.6 and Corollary 2.7.

4 A generalisation to certain multicones

4.1

The following result generalises, in part, Theorem 2.6. For a subset J of R, we denote by $J^\perp$ the set of roots orthogonal to J.

Theorem. *Let Q be a parabolic subgroup of G and let $y, w \in W^Q$. Let $I = \{\alpha \in \Delta \mid P_\alpha X_{wQ} = X_{wQ}$ and $P_\alpha C_{yQ} = C_{yQ}\}$. Suppose that there exist linearly independent positive roots $\beta_1, \ldots, \beta_q$ satisfying the following conditions:*

1) *For every $i = 1, \ldots, q$, $\alpha \in I$, and $a > 0$, $-\beta_i + a\alpha$ is not a root,*

2) *$X_{yQ} \subset X_{s_{\beta_i} yQ} \subseteq X_{wQ}$, for $i = 1, \ldots, q$,*

3) *$X_{wQ} = \overline{P_I U_{-\beta_1} \cdots U_{-\beta_q} X_{yQ}}$ and $\dim X_{wQ} = \dim X_{yQ} + q + \#(R_I^+ \setminus R_{I_0}^+)$, where $I_0 = I \cap \{\beta_1, \ldots, \beta_q\}^\perp$.*

Then:

(a) *$\mathcal{N}_{yQ,wQ}$ is L_I-isomorphic to $\mathcal{C}_I(\beta_1, \ldots, \beta_q)$, the L_I-orbit closure of the sum of highest weight vectors in the L_I-module $\bigoplus_{i=1,\ldots,q} V_I(-\beta_i)$. As a consequence, $N_{yQ,wQ}$ identifies with this module.*

(b) *Further, if* $\mathcal{C}_I(\beta_1,\ldots,\beta_q)\setminus\{0\}$ *is rationally smooth then one has*

$$P_{y,w} = \left(- \sum_{\substack{J\subseteq\{\beta_1,\ldots,\beta_q\}\\ J\neq\emptyset}} (q-1)^{|J|} \frac{H(W_I,q)}{H(W_{I\cap J^\perp},q)} \right)^{\leq(\ell(w)-\ell(y)-1)/2}.$$

Remarks. (i) The hypotheses of the theorem are satisfied, for instance, when $\beta_1,\ldots,\beta_q$ are pairwise orthogonal simple roots such that X_{yQ} is contained in each $X_{s_{\beta_i}yQ}$ and that $X_{wQ} = P_I X_{s_{\beta_1}\cdots s_{\beta_q}yQ}$. We will see in 4.3 that they are also satisfied for generic singularities of Schubert varieties in the variety of Lagrangian subspaces.

(ii) Hypothesis 3) can be weakened as $3)'$ $U_{-\beta_1}\cdots U_{-\beta_q}e_{yQ}\subset X_{wQ}$ and $\dim X_{wQ}\leq \dim X_{yQ}+q+\#(R_I^+\setminus R_{I_0}^+)$, as will be clear from the proof of the theorem. This formulation will be used in the proof of Proposition 4.4.

Proof. Hypothesis (2) implies that, for $i=1,\ldots,q$, the root subgroup $U_{-\beta_i}$ is contained in $U^-\cap y(U_Q^-)$. Together with hypothesis ($3'$), this implies that $U_{-\beta_1}\cdots U_{-\beta_q}e_{yQ}$ is contained in $(U^-\cap y(U_Q^-))e_{yQ}\cap X_{wQ} = \mathcal{N}_{yQ,wQ}$.

Now, let $u_i\in U_{-\beta_i}^\times$, for $i=1,\ldots,q$, and let $x=u_1\cdots u_q e_{yQ}$. Then $x\in\mathcal{N}_{yQ,wQ}$ and hence, being L_I-stable, $\mathcal{N}_{yQ,wQ}$ contains the orbit $L_I x$.

Let us compute the stabiliser $H=(L_I)_x$. First, for $\alpha\in I$, hypothesis (1) implies that U_α commutes with every $U_{-\beta_i}$ and hence $U_\alpha\subseteq H$. Since $U_I := L_I\cap U$ is generated by the U_α, $\alpha\in I$, it follows that $U_I\subseteq H$. Then, by Lemma 2.1, one deduces that H is generated by U_I, $H\cap T = \bigcap_{i=1,\ldots,q}\mathrm{Ker}(\beta_i)$, and the $U_{-\alpha}$ ($\alpha\in I$) that it contains. We claim that the latter are exactly the $U_{-\alpha}$ where $\alpha\in I_0$. Firstly, if $\alpha\in I_0$ then $U_{-\alpha}$ commutes with all $U_{-\beta_i}$ and fixes e_{yQ}, whence $U_{-\alpha}$ is contained in H. Secondly, by 2.1, again, H is normalised by T, and hence fixes all points of $\overline{Tx}$. Further, since the β_i are linearly independent, each $u_ie_{yQ}:=x_i$ belongs to $\overline{Tx}$ and hence H is contained in the isotropy group of each x_i. As in the proof of Proposition 2.3, this isotropy group is generated by U_I, $\mathrm{Ker}(\beta_i)$, and the $U_{-\alpha}$, for $\alpha\in I$ orthogonal to β_i. This concludes the proof of the claim.

Therefore, $\dim(L_Ix)=q+\#(R_I^+\setminus R_{I_0}^+)\geq\dim\mathcal{N}_{yQ,wQ}$ and L_Ix is open in $\mathcal{N}_{yQ,wQ}$. Further, the closure of Tx in $\mathcal{N}_{yQ,wQ}$ identifies with a T-module E with weights $-\beta_1,\ldots,-\beta_q$ of multiplicity 1. Set $P_0 := L_I\cap P_{I_0}$, and consider the natural morphism $\phi: L_I\times^{P_0}E\longrightarrow\mathcal{N}_{yQ,wQ}$, induced by the identification $E=Tx$ and the action of L_I on $\mathcal{N}_{yQ,wQ}$. Then, using the description of $(L_I)_x$ given above, one proves, similarly to 2.3, that ϕ is proper and birational. Since $\mathcal{N}_{yQ,wQ}$ is normal, it follows from Zariski's main theorem that $k[\mathcal{N}_{yQ,wQ}]\cong k[L_I\times^{P_0}E]$. Further, by 2.2($\dagger\dagger$), applied to L_I instead of G, the latter is isomorphic to $k[\mathcal{C}_I(\beta_1,\ldots,\beta_q)]$. Therefore,

$$k[\mathcal{N}_{yQ,wQ}] \cong k[\mathcal{C}_I(\beta_1,\ldots,\beta_q)].$$

Thus, since $\mathcal{N}_{yQ,wQ}$ and $\mathcal{C}_I(\beta_1,\ldots,\beta_q)$ are affine, they are isomorphic.

Now, set $\mathcal{C} = \mathcal{C}_I(\beta_1,\ldots,\beta_q)$ and suppose that $\mathcal{C} \setminus \{0\}$ is rationally smooth. Then so is $\mathcal{N}_{yB,wB} \setminus \{e_{yB}\}$, by assertion (a), coupled with Lemma 1.4. Thus, we may apply the argument of 1.5 to compute $P_{y,w}$. So, suppose that $\mathrm{char}(k) = p > 0$.

For $1 \leq i \leq q$, let v_i be a highest weight vector in $V_I(-\beta_i)$. For $J \subseteq \{\beta_1,\ldots,\beta_q\}$, let $v_J = \sum_{\beta_i \in J} v_i$, let $\mathcal{O}_J$ denote the L_I-orbit of v_J, and let $V_J = k$-span$\{v_i,\ \beta_i \in J\}$. Then the stabiliser of V_J in L_I is $L_I \cap P_{I \cap J^\perp}$ and hence, since the elements of J are linearly independent, $\mathcal{O}_J$ is a fibration over $L_I/(L_I \cap P_{I\cap J^\perp})$, with fiber $(k^\times)^{|J|}$. Therefore, the number of $\mathbb{F}_{p^r}$-rational points of $\mathcal{O}_J$ is $(p^r-1)^{|J|}\, H(W_I, p^r)\big/H(W_{I\cap J^\perp}, p^r)$. Since $\mathcal{C}\setminus\{0\}$ is the disjoint union of the $\mathcal{O}_J$, for $J \neq \emptyset$, assertion (b) then follows from Lemma 1.5(b).

4.2

Now, and until 4.5, we consider the case where $G = SP(2n)$ (the symplectic group in $GL(2n)$) and where Q is the stabiliser of a lagrangian subspace of k^{2n}. Then G/Q is not minuscule, but cominuscule (that is, Q is maximal and the associated simple root occurs in the highest root with coefficient one). We will then apply the previous result to describe the generic singularities of Schubert varieties in G/Q, the variety of lagrangian subspaces of k^{2n}.

The starting point is the following observation, which was pointed to us by V. Deodhar.

Lemma. *For cominuscule G/Q, the Bruhat order on G/Q is generated by the simple reflections.*

Proof. Let α be the simple root associated with Q. By assumption, α occurs in the highest root with coefficient 1. Therefore the fundamental weight $\omega_{\alpha^\vee} \in P(R^\vee)$, defined with respect to the base $\{\beta^\vee,\ \beta \in \Delta\}$ of $R^\vee$, is a minuscule weight. Further, under the natural identification $W(R) \cong W(R^\vee)$, the stabiliser of $\omega_{\alpha^\vee}$ in W equals W_Q. Thus, by [12, Lemma 1.14], applied to $(W(R^\vee), \omega_{\alpha^\vee})$, one obtains that the Bruhat order on W^Q is generated by the simple reflections.

4.3

For $w \in W^Q$, let us first describe $\mathrm{Bd}(X_{wQ})$, the boundary of X_{wQ} (see 3.2). We follow the notation of [2, Planche III] for the root system of type

C_n. In particular, $\Delta = \{\alpha_1, \dots, \alpha_n\}$, with α_n being the unique long root in Δ. Let $s_1, \dots, s_n$ denote the corresponding simple reflections.

Lemma. *Let $y < w$ in W^Q. Suppose that X_{yQ} is an irreducible component of $\mathrm{Bd}(X_{wQ})$. Then there exists a unique simple root β such that $X_{yQ} \subset X_{s_\beta yQ} \subseteq X_{wQ}$ and, denoting by I the union of the connected components of $\Delta \cap w(R^- \cup R_Q) \cap y(R_Q)$ to which β is adjacent, exactly one of the following possibilities holds.*

(1) *One has $X_{wQ} = P_I X_{s_\beta yQ}$, and either*

(1.a) *I is of type $A_r \times A_t$ and $I \cup \{\beta\}$ of type A_{r+t+1}, or*

(1.b) *I is of type A_r and $I \cup \{\beta\}$ of type C_{r+1}.*

(2) *One has $\beta = \alpha_m$, $I = \{\alpha_{m-r}, \dots, \alpha_{m-1}\} \cup \{\alpha_{m+1}, \dots, \alpha_{n-1}\}$, for some $r < m < n$, and $X_{wQ} = P_{\alpha_n} P_I X_{s_m yQ}$. In this case, $\ell(w) - \ell(y) = n - m + r + 1$.*

Proof. The proof is similar to that of Lemma 3.2. Let $J = \{\alpha \in \Delta \mid (w\omega, \alpha^\vee) \le 0\}$ and $I' = \{\alpha \in J \mid (y\omega, \alpha^\vee) = 0\}$. Then $P_J = \mathrm{Stab}_G(X_{wQ})$, and X_{yQ} is stable by P_J, since it is an irreducible component of $\mathrm{Bd}(X_{wQ}) = X_{wQ} \setminus P_J e_{wQ}$.

By Lemma 4.2, there exists $\beta \in \Delta$ such that $X_{yQ} \neq P_\beta X_{yQ} \subseteq X_{wQ}$. Then $(y\omega, \beta^\vee) > 0$ and, in particular, $\beta \notin J$. Let I be the union of the connected components of I' adjacent to β, and let $z = w_I s_\beta y$. Let us see whether X_{zQ} is P_J-stable. One has $z\omega = y\omega - (y\omega, \beta^\vee) w_I \beta$.

Let $\alpha \in I'$. Then $-w_I\alpha \in I'$ and hence $(\beta, -w_I\alpha^\vee) \le 0$. Therefore, $(z\omega, \alpha^\vee) \le 0$. It follows that X_{zQ} is stable by $P_{I'}$, and hence equals $P_I P_\beta X_{yQ}$.

Next, observe that for an arbitrary $\gamma \in R$, $(\omega, \gamma^\vee)$ belongs to $\{0, \pm 2\}$ if γ is short, and to $\{\pm 1\}$ if γ is long. In particular, $\alpha_n \notin I'$.

i) Suppose first that β is long, that is, $\beta = \alpha_n$. Then $(y\omega, \beta^\vee) = 1$. Let $\alpha \in J \setminus I'$. Since $\alpha \neq \beta$ then α is short and, since $(y\omega, \alpha^\vee) < 0$, one has $(y\omega, \alpha^\vee) = -2$. Since $(-w_I\beta, \alpha^\vee) \le 2$, it follows that $(z\omega, \alpha^\vee) \le 0$. This proves that X_{zQ} is P_J-stable. Since $X_{zQ} \not\subseteq \mathrm{Bd}(X_{wQ})$, it follows that $X_{wQ} = X_{zQ} = P_I X_{s_\beta yQ}$. Further, since $\beta \notin J$, this implies that $I \neq \emptyset$ and hence $I = \{\alpha_{n-r}, \dots, \alpha_{n-1}\}$ for some $r \ge 1$. This is case (1.b).

ii) Suppose now that β is short, say $\beta = \alpha_m$ for some $m < n$. Then $(y\omega, \beta^\vee) = 2$ and $(\beta, -w_I\alpha^\vee) \le 1$ for any $\alpha \in J$ ($\beta \neq -w_I\alpha$ since the latter is in R_J).

If α is a short root in $J \setminus I'$ then $(y\omega, \alpha^\vee) = -2$ and it follows that $(z\omega, \alpha^\vee) \le 0$. Next, suppose that $\alpha_n \in J \setminus I'$ and that $m < n-1$ and I has no connected component adjacent to α_n. Then $(w_I\beta, \alpha_n^\vee) = (\beta, \alpha_n^\vee) = 0$, and it follows that $(z\omega, \alpha^\vee) \le 0$ in this case.

Therefore, if $\alpha_n \notin J$ or in the case considered just above one obtains that $X_{wQ} = P_I X_{s_\beta yQ}$ and we are in the situation of Proposition 3.3, Case 1.

Thus, $I = \{\alpha_{m-r}, \dots, \alpha_{m-1}\} \cup \{\alpha_{m+1}, \dots, \alpha_{m+t}\}$ for some $1 \leq r < m$ and $1 \leq t < n - m$. This is the situation of case (1.a).

iii) Suppose finally that $\alpha_n \in J$, and that $m = n-1$ or I has a connected component adjacent to α_n. Then one has $I = \{\alpha_{m-r}, \dots, \alpha_{m-1}\} \cup \{\alpha_{m+1}, \dots, \alpha_{n-1}\}$. (If $r = 0$, resp. $m = n-1$, then the first, resp. second, set is empty).

In this case, one has $w_I\beta = \alpha_{m-r} + \cdots + \alpha_{n-1}$ and $(z\omega, \alpha_n^\vee) = 1$. Thus, X_{zQ} is not stable by P_{α_n}. Yet, one has $(s_n z\omega, \alpha_n^\vee) = -1$ and $(s_n z\omega, \alpha_\ell^\vee) = (z\omega, \alpha_\ell^\vee) \leq 0$, for $\alpha_\ell \in J \setminus \{\alpha_{n-1}, \alpha_n\}$. Further, one checks that $s_m w_I s_n \alpha_{n-1}^\vee = \alpha_m^\vee + \cdots + \alpha_{n-1}^\vee + 2\alpha_n^\vee$, and hence that $(s_n z\omega, \alpha_{n-1}^\vee) = 0$.

This proves that $X_{s_n zQ}$ is P_J-stable and hence equals X_{wQ}. Thus, $X_{wQ} = P_{\alpha_n} P_I X_{s_\beta yQ}$.

Then I_0, the set of roots in I orthogonal to $\beta = \alpha_m$, equals $\{\alpha_{m-r}, \dots, \alpha_{m-2}\} \cup \{\alpha_{m+2}, \dots, \alpha_{n-1}\}$, and one deduces that $w_I \equiv (s_{m-r} \cdots s_{m-1}) \cdot (s_{n-1} \cdots s_{m+1})$ modulo W_{I_0}. It follows that $\dim X_{zQ} - \dim X_{yQ} \leq r + n - m$. The equality could be proved by a direct argument, but since $X_{zQ} = P_I P_\beta X_{yQ}$ with I of type $A_r \times A_{n-1-m}$ and $I \cup \{\beta\}$ of type A_{r+n-m}, it follows from Proposition 3.3, Case 1), that $\dim X_{zQ} - \dim X_{yQ} = r + n - m$. Therefore, $\dim X_{wQ} - \dim X_{yQ} = r + n - m + 1$.

Moreover, one has $r \geq 1$. In fact, if $r = 0$ then $w\omega = s_n s_{n-1} \cdots s_m y\omega$ and hence $(y\omega, \alpha_m^\vee) = (w\omega, s_n \cdots s_m \alpha_m^\vee) = -(w\omega, \alpha_m^\vee + \cdots + \alpha_{n-1}^\vee + 2\alpha_n^\vee) = 0$, a contradiction. This shows that we are in case (2).

Finally, observe that in cases (1.a) and (1.b), resp. (2), β is uniquely determined by the equality $(y\omega, \beta^\vee)\beta = w_I(y\omega - w\omega)$, resp. $(y\omega, \beta^\vee)\beta = w_I(y\omega - s_n w\omega)$. This completes the proof of the proposition.

4.4

Proposition. *Let $w \in W^Q$. Then $\mathrm{Bd}(X_{wQ})$ is the singular locus of X_{wQ}. Indeed, if X_{yQ} is an irreducible component of $\mathrm{Bd}(X_{wQ})$, then (notation as in 4.3) :*

(a) *In case (1.a), $\mathcal{N}_{yQ,wQ}$ is isomorphic to the cone of decomposable tensors in $k^{n-m} \otimes k^{r+1}$, see* Proposition 3.3, Case 1.

(b) *In case (1.b), $N_{yQ,wQ} \cong S^2 k^{r+1}$ and $\mathcal{N}_{yQ,wQ}$ is isomorphic to the cone over the 2-uple embedding of $\mathbb{P}^r$ in $\mathbb{P}(S^2 k^{r+1})$. Therefore, one has $\mathrm{mult}_{yQ} X_{wQ} = 2^r$ and $P_{y,w} = 1$.*

(c) *In case (2), $\mathcal{N}_{yQ,wQ}$ is isomorphic to $\mathcal{C}$, the orbit closure of the sum of highest weight vectors in the $\mathrm{GL}(r+1) \times \mathrm{GL}(n-m)$-module*

$k^{r+1} \otimes k^{n-m} \oplus S^2 k^{r+1} = N_{yQ,wQ}$. *One has*

$$P_{y,w} = \sum_{i=0}^{\mathrm{Min}(r,n-m)} t^i \quad \textit{and} \quad \mathrm{mult}_{yQ} X_{wQ} = \sum_{i=0}^{r} \binom{n-m+r}{i}.$$

(c$'$) *Furthermore, in case* (2), $\mathcal{C}$ *identifies with the contraction to a point of the zero section of the vector bundle* $\mathcal{O}(-1) \otimes k^{n-m} \oplus \mathcal{O}(-2)$ *over* $\mathbb{P}^r$.

Proof. Let X_{yQ} be an irreducible component of $\mathrm{Bd}(X_{wQ})$. In cases (1.a) and (1.b), the assertions follow at once from Theorem 2.6 and Corollary 2.7. In these cases, e_{yQ} is a singular point of X_{wQ}.

Suppose now that we are in case (2). We saw in 4.3 (iii) that $w = s_{\alpha_n} w_I s_{\alpha_m} y$. Therefore, $w = w_I s_\beta s_\gamma y$, where $\beta = \alpha_m$ and $\gamma = s_\beta w_I \alpha_n = 2\sum_{i=m}^{n-1} \alpha_i + \alpha_n$. It is easily seen that β and γ satisfy hypothesis (1) of Theorem 4.1. We know already that $X_{yQ} \subset X_{s_\beta yQ} \subseteq X_{wQ}$. We claim that

$$X_{yQ} \subset X_{s_\gamma yQ} = X_{s_\beta s_\gamma yQ} \subseteq X_{wQ}. \qquad (*)$$

First, since X_{wQ} is P_I-stable, it is clear that $X_{wQ} \supseteq X_{s_\beta s_\gamma yQ}$. Further, one has $\gamma^\vee = \alpha_m^\vee + \cdots + \alpha_n^\vee$ and we saw in 4.3 (iii) that $(y\omega, \alpha_m^\vee) = 2$, $(y\omega, \alpha_n^\vee) = -1$ and $(y\omega, \alpha_i^\vee) = 0$ for $m < i < n$. Thus, $(y\omega, \gamma^\vee) = 1$ and hence $X_{yQ} \subset X_{s_\gamma yQ}$. Similarly, one checks that $s_\gamma \beta^\vee = -\alpha_m^\vee - 2\sum_{m<i\le n} \alpha_i^\vee$, so that $(s_\gamma y\omega, \beta^\vee) = 0$ and hence $X_{s_\gamma yQ} = X_{s_\beta s_\gamma yQ}$. This proves claim $(*)$.

One then deduces that $U_{-\gamma} e_{yQ}$ is contained in $X_{s_\gamma yQ}$, which is P_β-stable (recall that β is a simple root). It follows that $U_{-\beta} U_{-\gamma} e_{yQ} \subseteq X_{s_\gamma yQ} \subseteq X_{wQ}$.

Also, we saw in 4.3 (iii) that $\dim X_{wQ} - \dim X_{yQ} = r+n-m+1$ and that $I_0 = \{\alpha_{m-r}, \ldots, \alpha_{m-2}\} \cup \{\alpha_{m+2}, \ldots, \alpha_{n-1}\}$. Since, then, $\#(R_I^+ \setminus R_{I_0}^+) = r+n-1-m$, it follows that β, γ satisfy hypotheses (1),(2),(3$'$) of Theorem 4.1. Therefore, $\mathcal{N}_{yQ,wQ}$ is L_I-isomorphic to the orbit closure of the sum of highest weight vectors in the L_I-module $V_I(-\beta) \oplus V_I(-\gamma) = N_{yQ,wQ}$.

Further, by looking at the highest weights $-\beta$ and $-\gamma$, one sees that L_I acts on $N_{yQ,wQ}$ as $\mathrm{GL}(r+1) \times \mathrm{GL}(n-m)$ on $k^{r+1} \otimes k^{n-m} \oplus S^2 k^{r+1}$. This proves the first part of assertion (c).

Let us prove assertion (c$'$). Observe that $\mathcal{C} = \{u \otimes v \oplus tu^2 | u \in k^{r+1}, v \in k^{n-m}, t \in k\}$. Denote by $\widehat{\mathcal{C}}$ the subset of $\mathbb{P}^r \times \mathcal{C}$ consisting of all pairs $(x, u \otimes v \oplus tu^2)$ such that the point u lies on the line x. Then the first projection $p_1 : \widehat{\mathcal{C}} \to \mathbb{P}^r$ makes $\widehat{\mathcal{C}}$ the total space of the vector bundle $\mathcal{O}(-1) \otimes k^{n-m} \oplus \mathcal{O}(-2)$. Moreover, the second projection $p_2 : \widehat{\mathcal{C}} \to \mathcal{C}$ identifies $\mathcal{C}$ with the contraction to a point of the zero section of this vector bundle.

Now, let us prove the remaining part of assertion (c). First, using Lemma 1.5 and either of the descriptions of $\mathcal{C}$ given in (c) or (c′), one easily deduces that $P_{y,w}$ is as asserted. Secondly, $k[\mathcal{C}]$ is isomorphic to the bigraded algebra $\bigoplus_{i,j\geq 0} V_I(i(\omega_{m+1}+\omega_{m-1})+2j\omega_{m-1})$, and $\mathfrak{m}$, the maximal ideal corresponding to e_{yQ}, identifies with the augmentation ideal. Further, it follows from [14, Theorem 1.ii)] that one has $\mathfrak{m}^q = \bigoplus_{i+j\geq q} V_I(i(\omega_{m+1}+\omega_{m-1})+2j\omega_{m-1})$, for every $q \geq 1$. One deduces that

$$\mathfrak{m}^q/\mathfrak{m}^{q+1} \cong \bigoplus_{i=0}^{q} V_I\big(i\omega_{m+1}+(2q-i)\omega_{m-1}\big)$$

and, therefore, that

$$\dim\big(\mathfrak{m}^q/\mathfrak{m}^{q+1}\big) = \sum_{i=0}^{q} \binom{i+n-m-1}{n-m-1}\binom{2q-i+r}{r}.$$

It follows that

$$\mathrm{mult}_{yQ} X_{wQ} = \frac{(n-m+r)!}{(n-m-1)!\,r!}\,\kappa_{n-m-1,\,r},$$

where $\kappa_{a,b}$ denotes $\int_0^1 x^a(2-x)^b dx$, for $a,b\in\mathbb{N}$. Using integration by parts, one obtains that the $\kappa_{a,b}$ satisfy the recursion formula $(a+1)\kappa_{a,b} = 1+\kappa_{a+1,b-1}$. From this one deduces that, for all $a,b\in\mathbb{N}$, one has

$$\frac{(a+b+1)!}{a!\,b!}\,\kappa_{a,b} = \sum_{i=0}^{b}\binom{a+b+1}{i}.$$

This completes the proof of assertion (c). Finally, in all cases e_{yQ} is a singular point of X_{wQ}. This shows that $\mathrm{Bd}(X_{wQ})$ is the singular locus of X_{wQ}, as asserted.

4.5

The only other case of a cominuscule G/Q which is not minuscule is the case where $G = Spin(2n+1)$ and Q is the maximal parabolic corresponding to the fundamental weight ω_1 (the natural representation). But the results in this case are well-known and easily proved by direct arguments, as follows. Recall that G/Q is a smooth quadric hypersurface $\mathcal{Q}\subset\mathbb{P}(k^{2n+1})$. Moreover, each Schubert variety is the intersection of $\mathcal{Q}$ with a linear, B-stable subspace. But the B-stable subspaces of k^{2n+1} are: a flag of completely isotropic spaces $V_1,\dots,V_n$, their orthogonals $V_{n+1},\dots,V_{2n}$, and $V_{2n+1}=k^{2n+1}$ (indexed by their dimensions). It follows that the Schubert varieties in

G/Q are: the projective spaces $\mathbb{P}(V_1), \dots, \mathbb{P}(V_n) = \mathcal{Q} \cap \mathbb{P}(V_{n+1})$ and the quadratic cones $\mathcal{Q} \cap \mathbb{P}(V_{n+2}), \dots, \mathcal{Q} \cap \mathbb{P}(V_{2n+1})$. Denote by $X_0, \dots, X_{n-1}$ the former and by $X_n, \dots, X_{2n-1}$ the latter (indexed by their dimension). Clearly, $X_0, \dots, X_{n-1}$ and X_{2n-1} are smooth, but for $n \le i \le 2n-2$, X_i is singular along X_{2n-i-2} with a non-degenerate quadratic cone of dimension $2(i-n+1)$ as a transversal singularity. It follows that the multiplicity of X_i along X_{2n-i-2} is 2, whereas the corresponding Kazhdan–Lusztig polynomial is trivial.

4.6

As a final example, suppose now that $G = \mathrm{SL}(n{+}1)$, with $n \ge 3$, and consider the variety $F(1,n)$ of flags of type $(1,n)$ in k^{n+1}. Let $\{e_i,\ 1 \le i \le n{+}1\}$ be the standard basis of k^{n+1}. For $i = 0, \dots, n{+}1$, let $E_i = k$-span$\{e_q,\ q \le i\}$. It is easily seen that the Schubert varieties in $F(1,n)$ are exactly the

$$X_{i,j} = \{(\ell, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^* \mid \ell \subset H,\ \ell \subseteq E_i,\ E_{j-1} \subseteq H\},$$

for $1 \le i \ne j \le n{+}1$. Then, clearly, $X_{i,j}$ is smooth if $i < j$, or if $j = 1$ or $i = n{+}1$. So, suppose that $2 \le j < i \le n$. Then $X_{i,j}$ contains $X_{j-1,i+1}$ and one checks easily that $X_{i,j}$ is smooth outside $X_{j-1,i+1}$ and that the transversal along $X_{j-1,i+1}$ is isomorphic to

$$\{(x,y) \in E_i/E_{j-1} \times (E_i/E_{j-1})^* \mid \langle x, y\rangle = 0\},$$

which is a non-degenerate quadratic cone in $k^{2(i-j+1)}$. As we saw in Proposition 3.3, Case 3), the Kazhdan-Lusztig polynomial corresponding to this cone is $1 + t^{i-j}$.

Remark. The previous results could also be obtained by checking that Theorem 4.1 applies in that case.

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