Combinatorics of Reduced Words and Stanley symmetric functions

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Quotes

"Think deeply of simple things"

Arnold Ross

Outline

Reduced words of permutations

Stanley symmetric functions

Transition equations and the Little bump algorithm

Some properties characterized by permutation patterns

Reduced words of permutations

Generators. Every permutation $w \in S_\infty = \cup_{n>0} S_n$ can be written as a finite product of adjacent transpositions $s_i = (i, i+1)$, sometimes in many ways.

Example. My favorite permutation is

$$w = [2,1,5,4,3] = s_1 s_3 s_4 s_3 = s_1 s_4 s_3 s_4 = s_4 s_1 s_3 s_4 = s_4^3 s_1 s_3 s_4$$

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Relations.

- **Involution**: $s_i^2 = 1$ (identity permutation under multiplication)
- **Commutation**: $s_i s_j = s_j s_i$ provided $|i-j| > 1$,
- **Braid**: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Reduced words of permutations

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Defn.

- A minimal length expression for w is said to be *reduced*. This length $\ell(w) = inv(w) = \#\{i < j : w_i > w_j\}$.
- If $w = s_{a_1} \cdots s_{a_p}$ is reduced, then we say sequence $\mathbf{a} = (a_1, \ldots, a_p)$ is a *reduced word* for w.
- Let $R(w)$ be the set of *all reduced words* for w.

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- Let $R(w)$ be the set of *all reduced words* for w.

Example. If $w = [2, 1, 5, 4, 3]$ then $R(w)$ has 8 elements:

$$1343, 1434, 4134, 4314, 4341, 3431, 3413, 3143$$

Reduced words of permutations

Main Questions Today.

1. How can one count the number of reduced words of a permutation?
2. What sort of structure does this set have?
3. How does this relate to SYM and QSYM?

Reduced words of permutations

One approach to listing out all elements in $R(w)$:

Tits' Thm. The graph with vertices indexed by reduced words in $R(w)$ and edges connecting two words if they differ by a commutation move or a braid move is connected.

Example. If $w = [2, 1, 5, 4, 3]$ then $R(w)$ has 8 elements arranged in a cycle

1343 – 1434 – 4134 – 4314 – 4341 – 3431 – 3413 – 3143 – 1343

Example. How many reduced words does $w = [7, 1, 2, 3, 4, 5, 6]$ have?

Reduced words of permutations

A more efficient approach to counting $R(w)$:

Recurrence. There is one reduced word for $id = [1, 2, \ldots, n]$. For any other permutation w,

$$\#R(w) = \sum_{i \in Des(w)} \#R(ws_i).$$

Example. Compute $\#R([n, n-1, \ldots, 3, 2, 1])$ for small n:

$$1, 1, 2, 16, 768, 292864, 1100742656, 48608795688960,$$
$$29258366996258488320, 273035280663535522487992320, \ldots$$

Reduced words of Permutations

Stanley's First Observation. $\#R([n, n-1, \ldots, 3, 2, 1])$ for small values of n is the same as the number of standard tableaux of the staircase shape $(n-1, n-2, \ldots, 1)$ which is

$$\frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)^1}$$

by the hook length formula.

Question. How does $\binom{n}{2}$ relate to $w_0 = [n, n-1, \ldots, 3, 2, 1]$?

Compositions

Defn. A *composition* of a number p is a sequence of positive integers

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$$

such that $p = \sum \alpha_i = |\alpha|$.
Bijection: $Set : \{\alpha \vDash n\} \longrightarrow \{S \subset [p-1]\}$

$$\begin{aligned}(\alpha_1, \alpha_2, \ldots, \alpha_k) \longrightarrow \{&\alpha_1,\\ &\alpha_1 + \alpha_2,\\ &\alpha_1 + \alpha_2 + \alpha_3,\\ &\ldots\\ &\alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}\end{aligned}$$

Examples.
$Set(3, 2) = \{3\}$
$Set(1, 1, 4, 2) = \{1, 2, 6\}$

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Examples.
$Set(3, 2) = \{3\}$
$Set(1, 1, 4, 2) = \{1, 2, 6\}$
$Set^{-1}(\{2, 4\}) = ?$ Answer: Depends on p.

Fundamental basis for QSYM

Defn. Let α be a composition of p and $A = Set(\alpha) \subset [p-1] = \{1, 2, \ldots, p-1\}$.
The *fundamental quasisymmetric function*

$$F_\alpha = F_A^p(X) = \sum x_{i_1} \cdots x_{i_p}$$

summed over all $1 \leq i_1 \leq \ldots \leq i_p$ such that $i_j < i_{j+1}$ whenever $j \in A$.

Note: F_A^p may be abreviated F_A if the degree p is understood.

Examples.
$F_{(1,2)} = F^3_{\{1\}} = x_1x_2x_2 + x_1x_2x_3 + x_1x_3x_3 + x_2x_3x_3 + \ldots$
$= M_{(1,2)} + M_{(1,1,1)}$.

$F_{(1,2,1)} = F^4_{\{1,3\}} = x_1x_2x_2x_3 + x_1x_2x_2x_4 + x_1x_2x_3x_4 + \ldots$
$= M_{(1,2,1)} + M_{(1,1,1,1)}$.

Fundamental basis for QSYM

Poset on compositions. Given two compositions $\alpha = (\alpha_1, \ldots, \alpha_j)$ and $\beta = (\beta_1, \ldots, \beta_k)$ we say β *refines* *alpha* provided there exists indices $1 \leq a < b < \ldots < c \leq k$ such that $\alpha_1 = \beta_1 + \beta_2 + \ldots + \beta_a$ and $\alpha_2 = \beta_{a+1} + \ldots + \beta_b$, etc.

Write $\beta \preceq \alpha$ if β *refines* *alpha*.

Question. How does refinement order compare to the Boolean algebra on subsets?

Fundamental basis for QSYM

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Write $\beta \preceq \alpha$ if β *refines alpha*.

Question. How does refinement order compare to the Boolean algebra on subsets?

Lemma. $F_\alpha = \sum_{\beta \preceq \alpha} M_\beta$.

Example. $F_{(3,2)} = M_{(3,2)} + M_{(1,2,2)} + M_{(2,1,2)} + M_{(1,1,1,2)} + M_{(3,1,1)} + M_{(1,2,1,1)} + M_{(2,1,1,1)} + M_{(1,1,1,1,1)}$

Reduced words of Permutations

Defn. (Stanley 1984) For $w \in S_n$ with $p = inv(w)$, define

$$G_w(X) = \sum_{\mathbf{a} \in R(w)} F_{Des(\mathbf{a})}(X)$$
$$= \sum_{\mathbf{a} \in R(w)} \sum_{(i_1 \cdots i_p) \in C(\mathbf{a})} x_{i_1} x_{i_2} \cdots x_{i_p}$$

where $C(\mathbf{a})$ is the set of all weakly increasing sequences of positive integers $(i_1 \leq \cdots \leq i_p)$ such that $i_j \neq i_{j+1}$ if $a_j > a_{j+1}$.

$C(\mathbf{a})$ = *compatible sequences*

Reduced words of Permutations

$$G_w(X) = \sum_{\mathbf{a}\in R(w)} F_{Des(\mathbf{a})}(X) = \sum_{\mathbf{a}\in R(w)} \sum_{(i_1\cdots i_p)\in C(\mathbf{a})} x_{i_1} x_{i_2} \cdots x_{i_p}$$

Examples.

1. For $w = [7,1,2,3,4,5,6]$, $R(w) = \{654321\}$ so

$$G_w(X) = F_{\{1,2,3,4,5\}}(X) = F_{(1^6)}(X) = s_{(1^6)}(X).$$

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2. For $w = [2,1,5,4,3]$,
$$\begin{aligned} G_w =& F_{D(1343)} + F_{D(1434)} + F_{D(4134))} + F_{D(4314)} \\ &+ F_{D(4341)} + F_{D(3431)} + F_{D(3413)} + F_{D(3143)} = \\ F_{(3,1)}+&F_{(2,2)} + F_{(1,3)} + F_{(1,1,2)} + F_{(1,2,1)} + F_{(2,1,1)} + F_{(2,2)} + F_{(1,2,1)} \end{aligned}$$

Reduced words of Permutations

$$G_w(X) = \sum_{\mathbf{a}\in R(w)} F_{Des(\mathbf{a})}(X) = \sum_{\mathbf{a}\in R(w)} \sum_{(i_1\cdots i_p)\in C(\mathbf{a})} x_{i_1}x_{i_2}\cdots x_{i_p}$$

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2. For $w = [2,1,5,4,3]$,

$$G_w = F_{D(1343)} + F_{D(1434)} + F_{D(4134))} + F_{D(4314)} + F_{D(4341)} + F_{D(3431)} + F_{D(3413)} + F_{D(3143)} =$$

$$F_{(3,1)} + F_{(2,2)} + F_{(1,3)} + F_{(1,1,2)} + F_{(1,2,1)} + F_{(2,1,1)} + F_{(2,2)} + F_{(1,2,1)}$$

$$= s_{(3,1)} + s_{(2,1,1)} + s_{(2,2)}$$

Reduced words of Permutations

Stanley's Second Observation. For many permutations w, G_w is a symmetric function and it has a Schur positive expansion.

The fact that G_w is symmetric is proved in (Stanley 1984) by showing that every monomial x^α that occurs has the same coefficient as $x^{sort(\alpha)}$ by constructing explicit bijections on reduced words.

Schur positivity was originally more challenging.

Stanley symmetric functions

Thm.[Edelman-Greene 1987] G_w is symmetric and has Schur positive expansion:

$$G_w = \sum_{\mathbf{a} \in R(w)} F_{Des(\mathbf{a})} = \sum_{\lambda} a_{\lambda,w} s_\lambda, \qquad a_{\lambda,w} \in \mathbb{N}.$$

Cor. $|R(w)| = \sum_\lambda a_{\lambda,w} f^\lambda$ where f^λ is the number of standard tableaux of shape λ.

Nice cases.

1. If $w = [n, n-1, \ldots, 1] = w_0$ then $G_w = s_\delta$ where δ is the staircase shape with $n-1$ rows, hence $\#R(w_0) = f^\delta$.
2. $G_w = s_{\lambda(w)}$ iff w is 2143-avoiding iff w is *vexillary*.

Edelman-Greene Correspondence

Thm.[Edelman-Greene 1987] There exists an injective map from $R(w)$ to pairs of tableaux (P, Q) of the same shape where P is row and column strict and Q is standard. For each P every single standard tableaux of the same shape occurs as Q in the image.

Algorithm. Edelman-Greene insertion is a variation on RSK. The only difference is when inserting i into a row with i and $i+1$ already, skip that row and insert $i+1$ into the next row.

Example. For $1343 \in R([2,1,5,4,3])$,

$$\begin{array}{|c|}\hline 1\\\hline\end{array} \to \begin{array}{|c|c|}\hline 1&3\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&4\\\hline\end{array} \to \begin{array}{|c|c|c|}\cline{1-1} 4&\multicolumn{2}{c}{}\\\hline 1&3&4\\\hline\end{array}.$$

Edelman-Greene Correspondence

word	P	Q	$Des(Q)$
1343	$\begin{array}{lll} 4 & & \\ 1 & 3 & 4 \end{array}$	$\begin{array}{lll} 4 & & \\ 1 & 2 & 3 \end{array}$	$\{3\}$
1434	$\begin{array}{lll} 4 & & \\ 1 & 3 & 4 \end{array}$	$\begin{array}{lll} 3 & & \\ 1 & 2 & 4 \end{array}$	$\{2\}$
4134	$\begin{array}{lll} 4 & & \\ 1 & 3 & 4 \end{array}$	$\begin{array}{lll} 2 & & \\ 1 & 3 & 4 \end{array}$	$\{1\}$
3413	$\begin{array}{ll} 3 & 4 \\ 1 & 3 \end{array}$	$\begin{array}{ll} 3 & 4 \\ 1 & 2 \end{array}$	$\{2\}$
3143	$\begin{array}{ll} 3 & 4 \\ 1 & 3 \end{array}$	$\begin{array}{ll} 2 & 4 \\ 1 & 3 \end{array}$	$\{1, 3\}$

Edelman-Greene Correspondence

word	P	Q	$Des(Q)$
4314	4 3 1 4	3 2 1 4	$\{1, 2\}$
4341	4 3 1 4	4 2 1 3	$\{1, 3\}$
3431	4 3 1 4	4 3 1 2	$\{2, 3\}$

Edelman-Greene Correspondence

word	P	Q	$Des(Q)$
4314	$\begin{array}{\|c\|c\|} \hline 4 \\ \hline 3 \\ \hline 1 & 4 \\ \hline \end{array}$	$\begin{array}{\|c\|c\|} \hline 3 \\ \hline 2 \\ \hline 1 & 4 \\ \hline \end{array}$	$\{1,2\}$
4341	$\begin{array}{\|c\|c\|} \hline 4 \\ \hline 3 \\ \hline 1 & 4 \\ \hline \end{array}$	$\begin{array}{\|c\|c\|} \hline 4 \\ \hline 2 \\ \hline 1 & 3 \\ \hline \end{array}$	$\{1,3\}$
3431	$\begin{array}{\|c\|c\|} \hline 4 \\ \hline 3 \\ \hline 1 & 4 \\ \hline \end{array}$	$\begin{array}{\|c\|c\|} \hline 4 \\ \hline 3 \\ \hline 1 & 2 \\ \hline \end{array}$	$\{2,3\}$

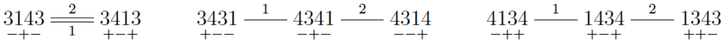
Observe. If $\mathbf{a}$ maps to (P, Q) under Edelman-Greene insertion, then $Des(\mathbf{a}) = Des(Q)$.

Properties of the Edelman-Greene Correspondence

Defn. The *Coxeter-Knuth graph* for w has vertices $R(w)$ and two words $a_1 \cdots a_p$ and $b_1 \cdots b_p$ are connected by an i-edge if they differ only in positions $i-1, i, i+1$ and on those positions they are of one of three forms

- **Witnessed Commutation**: $k\ i\ j \leftrightarrow i\ k\ j$ with $i < j < k$.
- **Witnessed Commutation**: $j\ k\ i \leftrightarrow j\ i\ k$ with $i < j < k$.
- **Braid**: $i\ (i+1) i = (i+1)\ i\ (i+1)$.

Example. Consider the Coxeter-Knuth graph for $[2, 1, 5, 4, 3]$



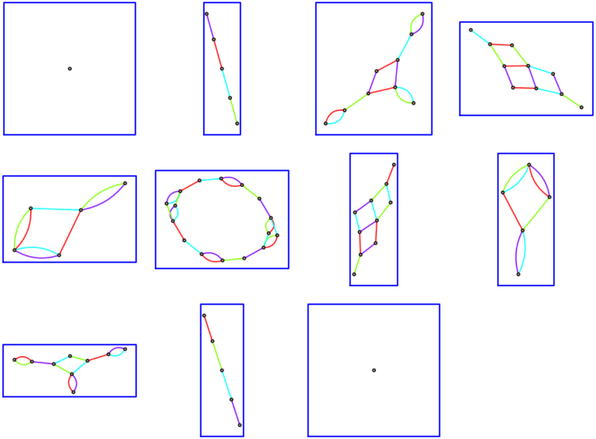
Properties of the Edelman-Greene Correspondence

Outline of the proof that the Stanley symmetric functions are Schur positive:

Thm.[Edelman-Greene 1987]

1. Two words $\mathbf{a}, \mathbf{b} \in R(w)$ have the same P-tableau if and only if they are in the same component of the Coxeter-Knuth graph for w.
2. If $EG(\mathbf{a}) = (P, Q)$ and Q' is another standard tableau of the same shape as Q, then $\mathbf{b} = EG^{-1}(P, Q')$ is Coxeter-Knuth equivalent to $\mathbf{a}$.
3. If $EG(\mathbf{a}) = (P, Q)$, then $Des(\mathbf{a}) = Des(Q)$.

All Coxeter-Knuth Graphs for Length 6 Words



Coxeter-Knuth Graphs $\approx$ Dual Equivalence Graphs

Defn. The Coxeter-Knuth graph for w has $V = R(w)$ and two reduced words are connected by an edge labeled i if they agree in all positions except for a single Coxeter-Knuth relation starting in position i.

Defn. (Assaf, 2008) Dual equivalence graphs are graphs with labeled edges whose connected components are isomorphic to the graph on standard tableaux of a fixed partition shape with an edge labeled i connecting any two vertices which differ by a transposition (i,i+1) or (i+1,i+2) with the third number on a diagonal in between the transposing pair.

Coxeter-Knuth Graphs and Dual Equivalence

Thm. The Coxeter-Knuth graphs in type A are dual equivalence graphs and the isomorphism is given by the Q tableaux in Edelman-Greene insertion. Furthermore, descent sets are preserved.

In type A, this is a nice corollary of (Roberts, 2014) $+$ (Hamaker-Young, 2014).

Thm.(Chmutov, 2013$+$) Stembridge's A-molecules are dual equivalence graphs and the edge labeling comes from labeling the Coxeter graph's edges consecutively.

Transition Equation

Notation. Let $1 \times w = [1, w_1 + 1, w_2 + 1, \ldots, w_n + 1]$. There is a bijection from $R(w)$ to $R(1 \times w)$ that preserves descent sets, so $G_w = G_{1\times w}$.

Thm.[Lascoux-Schützenberger] If w is vexillary, then $G_w = s_{\lambda(w)}$. Otherwise, let $(r < s)$ be the lexicographic largest pair of values inverted in w, then

$$G_w = \sum G_{w'}$$

where the sum is over all w' such that $inv(w) = inv(w')$ and $w' = t_{ir} t_{rs} w$ with $0 < i < r$. Call this set $T(w)$. In the case $T(w)$ is empty, replace w by $1 \times w$.

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where the sum is over all w' such that $inv(w) = inv(w')$ and $w' = t_{ir} t_{rs} w$ with $0 < i < r$. Call this set $T(w)$. In the case $T(w)$ is empty, replace w by $1 \times w$.

Example. If $w = [6, 3, 2, 7, 4, 5, 1]$, then $r = 5$, $s = 7$

$$G_{[6,3,2,7,4,5,1]} = G_{[6,3,5,2,4,7,1]} + G_{[6,5,2,3,4,7,1]}$$

So, $T(w) = \{[6, 3, 5, 2, 4, 7, 1], [6, 5, 2, 3, 4, 7, 1]\}$.

Transition Tree

We can make a tree starting with w as the root and the children of a node v will be the permutations in $T(v)$ or the empty set if v is vexillary (could keep going until v^{-1} has only 1 descent).

Example.

$$[3, 2, 1, 4, 6, 5]$$

$$|$$

$$[3, 2, 1, 5, 4, 6]$$

$$/ \qquad | \qquad \backslash$$

$$[3, 2, 4, 1, 5, 6] \quad [3, 4, 1, 2, 5, 6] \quad [4, 2, 1, 3, 5, 6]$$

Transition Equation

Thm.[Lascoux-Schützenberger 1982] If w is vexillary, then $G_w = s_{\lambda(w)}$. Otherwise,

$$G_w = \sum G_{w'}$$

where the sum is over all w' such that $l(w) = l(w')$ and $w' = t_{ir} t_{rs} w$ with $0 < i < r$. Call this set $T(w)$. In the case $T(w)$ is empty, replace w by $1 \times w$. This algorithm terminates.

Cor. If w is vexillary, $\#R(w) = f^{\lambda(w)}$, otherwise

$$\#R(w) = \sum_{w' \in T(w)} \#R(w')$$

Transition Equation

Question. Is there a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the descent set, Coxeter-Knuth classes and the Q tableau of each reduced word?

A bijection preserving descent sets alone would prove the Stanley symmetric functions are Schur positive provided we can show this holds for vexillary permutations.

Little's Bijection

Question. Is there a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the descent set, Coxeter-Knuth classes and the Q tableau?

Answer. Yes! It's called Little's bijection named for David Little (Little, 2003) + (Hamaker-Young, 2014).

Thomas Lam's Conjecture.(proved by Hamaker-Young, 2014) Every reduced word for any permutation with the same Q tableau is connected via Little bumps. Every communication class under Little bumps contains a unique reduced word for a unique minimal inverse Grassmannian permutation.

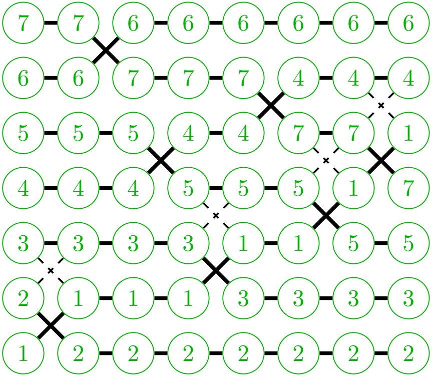
The Little Bump Algorithm

Given a reduced word, there is an associated reduced wiring diagram. If removing a crossing leaves another reduced wiring diagram, that crossing is a candidate to initiate a Little bump by pushing it down. Pushing down means reduce the corresponding letter in the word by 1.

- Check if the resulting word is reduced. If so, stop and return the new word.
- Otherwise, find the other point where the same two wires cross, and push that crossing down in the same direction. Repeat previous step.

Little Bijection. Initiate a Little bump at the crossing (r, s) corresponding to the lex largest inversion.

The Algorithm in Pictures



Reduced words of permutations

Review of Main Questions Today.

1. How can one count the number of reduced words of a permutation?

2. What sort of structure does this set have?

3. How does this relate to SYM and QSYM?

Review Reduced words of permutations

Main Questions Today.

1. How can one count the number of reduced words of a permutation? Answer: Three ways: partition words by last letter, use Edelman-Greene tableaux, use the transition equations.

2. What sort of structure does this set have? Answer: Coxeter-Knuth graphs, Edelman-Greene correspondance, Little bumps and bijections.

3. How does this relate to SYM and QSYM? Answer: Stanley symmetric functions.

Curious Application of the Little Bijection

Thm.(Macdonald 1991, Fomin-Stanley 1994, Young 2014) For $w_0 = [n, n-1, \ldots, 2, 1]$,

$$\sum_{a_1 \ldots a_p \in R(w_0)} a_1 a_2 \cdots a_p = \binom{n}{2}!$$

Thm.(Young 2014) There exists an algorithm based on the Little bijection to choose a reduced word $\mathbf{a} = a_1 \ldots a_p \in R(w_0)$ with probability distribution

$$P(\mathbf{a}) = \frac{a_1 \cdots a_2 \cdots a_p}{\binom{n}{2}!}$$

Compare to "Random Sorting Networks" by Angel-Holroyd-Romik-Virag 2007.

Open Problem

Thm.(Macdonald 1991, Fomin-Stanley 1994) For any permutation w of length p

$$\sum_{a_1 \ldots a_p \in R(w)} a_1 a_2 \cdots a_p = p! \cdot \mathfrak{S}_w(1, 1, \ldots, 1)$$

where $\mathfrak{S}_w(x_1, \ldots, x_n)$ is the Schubert polynomial for w.

Open Problem. Find a bijective proof.

Vexillary Permutations

Def. A permutation is *vexillary* iff $G_w = s_{\lambda(w)}$ iff w is 2143-avoiding.

Properties.

- Schubert polynomial is a flagged Schur function (Wachs).
- Kazhdan-Lusztig polynomials have a combinatorial formula (Lascoux-Schützenberger).
- Nice enumeration, the same as 1234-avoiding permutations (Gessel,West).
- Easy to find a uniformly random reduced expression using Edelman-Greene correspondence and the hook-walk algorithm (Greene-Nijenhuis-Wilf).

Generalizing Vexillary Permutations

Def. A permutation is *k-vexillary* iff $G_w = \sum a_{\lambda,w} s_\lambda$ and $\sum a_{\lambda,w} \leq k$.

Example. $G_{[2,1,4,3,6,5]} = S_{(3)} + 2S_{(2,1)} + S_{(1,1,1)}$
so $[2, 1, 4, 3, 6, 5]$ is 4-vexillary, but not 3-vexillary.

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Thm. (Billey-Pawlowski) A permutation w is k-vexillary iff w avoids a finite set of patterns V_k for all $k \in \mathbb{N}$.

$k = 1$ $V_1 = \{2143\}$,
$k = 2$ $|V_2| = 35$, all in $S_5 \cup S_6 \cup S_7 \cup S_8$
$k = 3$ $|V_3| = 91$, all in $S_5 \cup S_6 \cup S_7 \cup S_8$
$k = 4$ $|V_4| = 2346$, all in $S_5 \cup \cdots \cup S_{12}$
($k = 4$ case required help from Michael Albert)

Generalizing Vexillary Permutations

Def. A permutation is *k-vexillary* iff $G_w = \sum a_{\lambda,w} s_\lambda$ and $\sum a_{\lambda,w} \leq k$.

Properties.

- 2-vex perms have easy expansion: $G_w = s_{\lambda(w)} + s_{\lambda(w^{-1})'}$.
- 3-vex perms are multiplicity free: $G_w = s_{\lambda(w)} + s_\mu + s_{\lambda(w^{-1})'}$ for some μ between first and second shape in dominance order.
- 3-vex perms have a nice essential set.

Outline of Proof

Thm. (Billey-Pawlowski) A permutation w is k-vexillary iff w avoids a finite set of patterns V_k for all $k \in \mathbb{N}$.

Proof.

1. (James-Peel) Use generalized Specht modules S^D for $D \in \mathbb{N} \times \mathbb{N}$.
2. (Kraśkiewicz, Reiner-Shimozono) For $D(w)$=diagram of permutation w,
$$S^{D(w)} = \bigoplus (S^\lambda)^{a_{\lambda,w}}.$$
3. Compare Lascoux-Schützenberger transition tree and James-Peel moves.
4. If w contains v as a pattern, then the James-Peel moves used to expand $S^{D(v)}$ into irreducibles will also apply to $D(w)$ in a way that respects shape inclusion and multiplicity.

Another permutation filtration

Def. A permutation w is *multiplicity free* if G_w has a multiplicity free Schur expansion.

Def. A permutation w is *k-multiplicity bounded* if $\langle G_w, S_\lambda \rangle \leq k$ for all partitions λ.

Cor. If w is k-multiplicity bounded and w contains v as a pattern, then v is k-multiplicity bounded for all k.

Motivation

Let $D \subset \mathbb{N} \times \mathbb{N}$. Let $S^D = \bigoplus (S^\lambda)^{c_{\lambda,D}}$ expanded into irreducibles.

In the Grassmannian $Gr(k, n)$, consider the row spans of the matrices

$$\{(I_k|A) : A \in M_{k\times(n-k)}, A_{ij} = 0 \text{ if } (i,j) \in D\}.$$

Let Ω_D be the closure of this set in $Gr(k, n)$. Let σ_D be the cohomology class associated to this variety.

Liu's Conjecture. The Schur expansion of $\sigma_D = \sum c_{\lambda,D} S_\lambda$.

True for "forests" (Liu 2009), not true for permutation diagrams (Pawlowski 2014).

Future Work

Conjecture.(Billey-Pawlowski) The multiplicity free permutations are characterized by 198 patterns up through S_{11}.

Question. What other properties of Stanley symmetric functions are characterized by permutation pattern avoidance?