A $1,884,500,000(CP) open problem relating Schur-positivity and F-positivity

Sara Billey
University of Washington
http://www.math.washington.edu/~billey/colombia/

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Quote

"What problems have you worked on and failed to solve?"

Lenore Cowen (around 2000)

Outline

Review of SYM and QSYM notation

Open Problem: Plethysm of Schur Functions

Motivation: P vs NP

One New Approach using QSYM: Loehr-Warrington Theorem

Overview

SYM	QSYM
nice bases: $e_\lambda, h_\lambda, m_\lambda, p_\lambda, s_\lambda$	nice bases: $M_\alpha, F_\alpha, \mathfrak{S}^*_\alpha, \mathcal{S}_\alpha, N_\alpha$
indexed by partitions	indexed by compositions
rep theory of S_n and GL_n	rep theory of 0-Hecke algebra
$m_\lambda = \sum_{sort(\alpha)=\lambda} M_\alpha$	$F_\alpha = \sum_{\beta \preceq \alpha} M_\beta$
$s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu$	$s_\lambda = \sum_{T \in SYT(\lambda)} F_{D(T)}$

Background on Plethysm

Defn. Given two symmetric functions $f(x_1, x_2, \dots)$ and $g(x_1, x_2, \dots) = x^a + x^b + x^c + \dots$ define the *plethysm* of f and g to be the function

$$f[g] = f(x^a, x^b, x^c, \dots).$$

Then, $f[g]$ is again a symmetric function.

Example. Expand $h_2[e_2(X)]$ on $X = \{x_1, x_2, x_3\}$

$$\begin{aligned} h_2[e_2(x_1, x_2, x_3)] &= h_2[x_1x_2 + x_1x_3 + x_2x_3] \\ &= h_2[x_1x_2, x_1x_3, x_2x_3] \end{aligned}$$

$$= (x_1x_2)^2 + (x_1x_3)^2 + (x_2x_3)^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2$$

$$= s_{(2,2)}(x_1, x_2, x_3).$$

Computing Plethysm with Sage

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```
Sym = SymmetricFunctions(QQ)
s = Sym.schur()
e = Sym.elementary()
h = Sym.complete()
s[2].plethysm(s[1,1])
 s[1, 1, 1, 1] + s[2, 2]

s[2].plethysm(s[3])
 s[4, 2] + s[6]

s[2].plethysm(s[4])
 s[4, 4] + s[6, 2] + s[8]

s[2].plethysm(s[5])
 s[6, 4] + s[8, 2] + s[10]

s[2].plethysm(s[3,1])
 s[3, 3, 1, 1] + s[4, 2, 2] + s[4, 3, 1] + s[4, 4] + s[5, 1, 1, 1] + s[5, 2, 1] + s[6, 2]


latex(s[2,2].plethysm(s[3,1]))
 s_{3,3,3,3,2,2} + s_{4,3,3,2,2,1,1} + s_{4,3,3,3,1,1,1} + 2s_{4,3,3,3,2,1} + s_{4,3,3,3,3} + s_{4,4,2,2,1,1,1,1} + s
 1} + 3s_{4,4,3,2,2,1} + 5s_{4,4,3,3,1,1} + 2s_{4,4,3,3,2} + s_{4,4,4,1,1,1,1} + 2s_{4,4,4,2,1,1} + 4s_{4,4,4,2,2} +
 _{5,3,2,2,2,1,1} + s_{5,3,3,1,1,1,1,1} + 3s_{5,3,3,2,1,1,1} + 5s_{5,3,3,2,2,1} + 3s_{5,3,3,3,1,1} + 5s_{5,3,3,3,2} +
 ,2,1} + 4s_{5,4,3,1,1,1,1} + 14s_{5,4,3,2,1,1} + 10s_{5,4,3,2,2} + 12s_{5,4,3,3,1} + 7s_{5,4,4,1,1,1} + 13s_{5,4,4,
 ,1,1,1} + 2s_{5,5,2,1,1,1,1} + 9s_{5,5,2,2,1,1} + 2s_{5,5,2,2,2} + 6s_{5,5,3,1,1,1} + 16s_{5,5,3,2,1} + 9s_{5,5,3,3
 + 4s_{5,5,5,1} + s_{6,2,2,2,2,2} + s_{6,3,2,1,1,1,1,1} + 3s_{6,3,2,2,1,1,1} + 4s_{6,3,2,2,2,1} + 3s_{6,3,3,1,1,1,1
 2,2} + 8s_{6,3,3,3,1} + 6s_{6,4,2,1,1,1,1} + 11s_{6,4,2,2,1,1} + 11s_{6,4,2,2,2} + 17s_{6,4,3,1,1,1} + 31s_{6,4,3,2
 1} + 17s_{6,4,4,2} + 2s_{6,5,1,1,1,1,1} + 13s_{6,5,2,1,1,1} + 18s_{6,5,2,2,1} + 25s_{6,5,3,1,1} + 21s_{6,5,3,2} + 1
 6,1,1,1,1} + 9s_{6,6,2,1,1} + 12s_{6,6,2,2} + 10s_{6,6,3,1} + 6s_{6,6,4} + s_{7,2,2,2,1,1,1} + s_{7,2,2,2,2,1} + s_
```

Open Problem

Fact. For any two partitions μ, ν, the plethysm $s_\mu[s_\nu]$ has a Schur positive expansion.

Reason. $s_\mu[s_\nu]$ is the Frobenius characteristic of an S_{ab} representation if $\mu \vdash a$ and $\nu \vdash b$.

Open Problem. Find a combinatorial/optimal formula for the coefficients $d^\lambda_{\mu,\nu}$ in the expansion

$$s_\mu[s_\nu] = \sum_\lambda d^\lambda_{\mu,\nu} s_\lambda$$

.

Special Case of Plethysm

Thm.(Thrall 1942) For $X = \{x_1, x_2, \dots\}$

$$s_{(2)}[s_{(n)}(X)] = h_2[h_n(X)] = \sum_{k=0}^{\lfloor n/2 \rfloor} s_{(2n-2k,2k)}(X)$$

Examples.

$s_{(2)}[s_{(3)}] = s_6 + s_{4,2}$

$s_{(2)}[s_{(4)}] = s_8 + s_{6,2} + s_{4,4}$

$s_{(2)}[s_{(5)}] = s_{10} + s_{8,2} + s_{6,4}$

Motivation

P = set of all "yes/no" questions which can be decided in polynomial time depending on the input size.

NP = set of all "yes/no" questions for which one can test a proposed solution in polynomial time depending on the input size.

#**P** = set of all questions of the form "How many solutions does X have?" where X is in **NP**.

Examples.

1. Does a graph G have a planar embedding? $\in$ **P** (Kuratowski 1930, Hopcroft-Tarjan 1974)
2. Does G have a 3-coloring? $\in$ **NP** (Garey-Johnson-Stockmeyer 1976)
3. How many k-colorings does G have for $k = 1, 2, 3, \ldots$? $\in$ #**P**
4. What are the coefficients of the chromatic polynomial? (Jaeger-Vertifan-Welsh 1990)

Motivation

P = set of all "yes/no" questions which can be decided in polynomial time depending on the input size.

NP = set of all "yes/no" questions for which one can test a proposed solution in polynomial time depending on the input size.

#**P** = set of all questions of the form "How many solutions does X have?" where X is in **NP**.

Examples.

1. Does a permutation $w \in S_n$ contain another $v \in S_k$? (Bose-Buss-Lubiw, 1998)
2. How many instances of the permutation $v \in S_k$ does $w \in S_n$ contain? (Bose-Buss-Lubiw, 1998)
3. Does a permutation $w \in S_n$ contain a fixed $v \in S_k$ (Guillemot-Marx, 2013)

Motivation

P $=$ set of all questions which can be decided in polynomial time depending on the input size.

NP $=$ set of all questions for which one can test a proposed solution in polynomial time depending on the input size.

#**P** $=$ set of all questions of the form "How many solutions does X have?" where X is in **NP**.

Examples.

1. What is the determinant of an $n \times n$ matrix? (Williams 2012, Cohn-Umans 2013)
2. What is the permanent of an $n \times n$ matrix? (Valiant 1979)

Motivation

P $=$ set of all questions which can be decided in polynomial time depending on the input size.

NP $=$ set of all questions for which one can test a proposed solution in polynomial time depending on the input size.

$\#$**P** $=$ set of all questions of the form "How many solutions does X have?" where X is in **NP**.

Open. Does **P** $=$ **NP**? Does **NP** $=$ $\#$**P**? Does **P** $=$ $\#$**P**?

Clay Millennium Prize:$1,000,000 in US dollars for "P vs NP" problem.

Motivation

Mulmuley-Sohoni (2001-present) approach to "$P \neq NP$":

1. Homogeneous degree n polynomials form a vector space of dimension N with a GL_N action, in addition to a GL_n action.
2. The determinant of an $n \times n$ matrix is a homogeneous polynomial of degree n^2 which is computable in $O(n^3)$ time, perhaps $O(n^{2+\epsilon})$ (Cohn-Kleinberg-Szegedy-Umans 2005)
3. The permanent of an $n \times n$ matrix is a homogeneous polynomial degree n^2. Its computation is a #P-complete problem (Valiant, 1979a).
4. Every formula f of size u can be written as a determinant of some $k \times k$ matrix M_f with entries depending linearly on the original inputs where $k \leq 2u$. (Valiant, 1979b)
5. Use GL_N representation theory to study the orbit of the permanent vs determinant. In particular, they relate it to decomposing $V^\mu(V^\nu)$ where V^μ, V^ν are irreducible GL_N reps.

Plethysm and QSYM

Thm.(Loehr-Warrington 2012) For any two partitions μ, ν

$$s_\mu[s_\nu(X)] = \sum_{A \in S_{a,b}(\mu,\nu)} F_{Des(rw(A)^{-1})}.$$

where $S_{a,b}(\mu, \nu)$ is a set of $a \times b$-matrices with positive integer entries and $Des(rw(A)^{-1})$ is the descent set of a permutation associated to A.

Example.

$$s_{(2)}[s_{(3)}] = s_{(4,2)} + s_{(6)} = F_{[1,2,3]} + F_{[1,3,2]} + F_{[1,4,1]} + F_{[2,2,2]} + F_{[2,3,1]} + F_{[2,4]} + F_{[3,2,1]} + F_{[3,3]} + F_{[4,2]} + F_{[6]}$$

Concrete Notation

The plethysm $s_\mu[s_\nu]$ is the generating function for column strict tableaux with entries which are column strict tableaux.

For $\mu = (2,2)$ and $\nu = (3,2,1)$, such a tableau could be

$$V = \begin{array}{|c|c|} \hline T = \begin{array}{|c|c|c|} \hline 6 & & \\ \hline 3 & 3 & \\ \hline 2 & 2 & 2 \\ \hline \end{array} & U = \begin{array}{|c|c|c|} \hline 7 & & \\ \hline 3 & 5 & \\ \hline 2 & 4 & 4 \\ \hline \end{array} \\ \hline S = \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 2 & 3 & \\ \hline 1 & 1 & 2 \\ \hline \end{array} & S = \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 2 & 3 & \\ \hline 1 & 1 & 2 \\ \hline \end{array} \\ \hline \end{array}$$

The *weight* of such a tableau is the product of the weights of each entry. So $wt(V) = x^S x^S x^T x^V = x_1^4 x_2^8 x_3^5 x_4^4 x_5 x_6 x_7$.

Concrete Notation

$SSYT(\nu)$ = set of column strict (SemiStandard Young) Tableaux of shape ν

If ν is a fixed partition shape, then we can identify $T \in SSYT(\nu)$ with its (Spanish) *reading word*.

$$T = \begin{array}{|c|c|c|} \hline 6 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 1 & 2 \\ \hline \end{array} \longrightarrow rw(T) = 623112$$

$\mathcal{W}(\nu) = \{rw(T) \mid T \in SSYT(\nu)\}$ ordered lexicographically

Concrete Notation

For $\mu = (2,2)$ and $\nu = (3,2,1)$,

$$V = \begin{array}{|c|c|} \hline T = \begin{array}{|c|c|c|} \hline 6 & & \\ \hline 3 & 3 & \\ \hline 2 & 2 & 2 \\ \hline \end{array} & U = \begin{array}{|c|c|c|} \hline 7 & & \\ \hline 3 & 5 & \\ \hline 2 & 4 & 4 \\ \hline \end{array} \\ \hline S = \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 2 & 3 & \\ \hline 1 & 1 & 2 \\ \hline \end{array} & S = \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 2 & 3 & \\ \hline 1 & 1 & 2 \\ \hline \end{array} \\ \hline \end{array}$$

maps to

$$V' = \begin{array}{|c|c|} \hline 633222 & 735244 \\ \hline 423112 & 423112 \\ \hline \end{array}$$

$SSYT_{\mathcal{W}(\nu)}(\mu) =$ set of all $SSYT(\mu)$ with entries in $\mathcal{W}(\nu)$

Concrete Notation

For $\mu = (2,2)$ and $\nu = (3,2,1)$,

$$V = \begin{array}{|c|c|}\hline T = \begin{array}{|c|cc}\hline 6 & & \\ \hline 3 & \multicolumn{1}{|c|}{3} & \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} & U = \begin{array}{|c|cc}\hline 7 & & \\ \hline 3 & \multicolumn{1}{|c|}{5} & \\ \hline 2 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \\ \hline S = \begin{array}{|c|cc}\hline 4 & & \\ \hline 2 & \multicolumn{1}{|c|}{3} & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} & S = \begin{array}{|c|cc}\hline 4 & & \\ \hline 2 & \multicolumn{1}{|c|}{3} & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \\ \hline \end{array}$$

maps to

$$V' = \begin{array}{|c|c|}\hline 633222 & 735244 \\ \hline 423112 & 423112 \\ \hline \end{array}$$

maps to

$$V'' = \begin{pmatrix} 6 & 3 & 3 & 2 & 2 & 2 \\ 7 & 3 & 5 & 2 & 4 & 4 \\ 4 & 2 & 3 & 1 & 1 & 2 \\ 4 & 2 & 3 & 1 & 1 & 2 \end{pmatrix}$$

$M(\mu, \nu) =$ matrices obtained from $SSYT(\mu)$ with entries in $\mathcal{W}(\nu)$.

Recap

$M(\mu, \nu) =$ matrices obtained from $SSYT(\mu)$ with entries in $\mathcal{W}(\nu)$.

Lemma. (L-W)

$$s_\mu[s_\nu] = \sum_{A \in M(\mu,\nu)} wt(A)$$

where $wt(A) = \prod_{i,j} x_{A(i,j)}$.

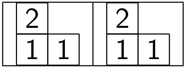
That's a big sum of monomials! How do we collect terms?

Which basis would give us the most compression while being reasonably easy to prove?

Standardization of integer matrices

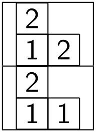
False Start. Standardize each matrix in $M(\mu, \nu)$ by standardizing the reading word of the matrix in the usual Spanish reading order.

If $\mu = (2)$ and $\nu = (2,1)$,



$\longrightarrow$ | 211 | 211 | $\longrightarrow \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 5 & 1 & 2 \\ 6 & 3 & 4 \end{pmatrix}$

If $\mu = (1,1)$ and $\nu = (2,1)$,



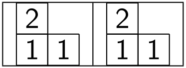
$\longrightarrow$ | 212 / 211 | $\longrightarrow \begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 1 & 5 \\ 6 & 2 & 3 \end{pmatrix} \notin M(\mu, \nu)$

Standardization of integer matrices

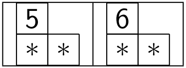
False Start. Standardize each matrix in $M(\mu,\nu)$ by standardizing the reading word of the matrix in the usual Spanish reading order.

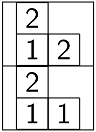
Problem. That does not preserve the column strict property on tableaux containing tableaux.

Try again. What should happen on small cases ?

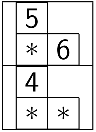


$$\longrightarrow \begin{pmatrix}2 & 1 & 1\\ 2 & 1 & 1\end{pmatrix} \longrightarrow \begin{pmatrix}5 & * & *\\ 6 & * & *\end{pmatrix} \longrightarrow$$





$$\longrightarrow \begin{pmatrix}2 & 1 & 2\\ 2 & 1 & 1\end{pmatrix} \longrightarrow \begin{pmatrix}5 & * & 6\\ 4 & * & *\end{pmatrix} \longrightarrow$$



$* = 2, 3$

Standardization of integer matrices

$M_{a,b}$ = all $a \times b$ matrices with entries in $\mathbb{P}$= *positive matrices*
$S_{a,b} \subset M_{a,b}$, entries are exactly $\{1, 2, \ldots, ab\}$=*standard matrices*

Given a matrix $A \in S_{a,b}$, define the *reading word* $rw(A)$ in a new way: read down the last column, this becomes the last a letters of the word. Next read the second to last column, in the order given by the last column, this becomes the second to last a letters of the word, etc.

$$A = \begin{pmatrix} 8 & 7 & 2 & \underline{10} \\ 6 & 1 & 9 & 4 \\ \underline{12} & 5 & \underline{11} & 3 \end{pmatrix} \in S_{3,4} \qquad rw(A) = 6\,\underline{12}\,8\,.\,7\,1\,5\,.\,\underline{11}\,9\,2\,.\,\underline{10}\,4\,3$$

If $rw(B) = 4\ \underline{12}\ 8.\ 6\ 9\ \underline{10}.\ 5\ 2\ 7.\ 3\ \underline{11}\ 1$, what is $B \in S_{3,4}$?

Standardization of integer matrices

Given a matrix $M \in M_{a,b}$, define the *standardization* to be $S(M) = (std(M), sort(M))$ where $std(M)$ is given by the algorithm:

- For $k \geq 0$, let $N(k) = \#\{M_{i,j} \leq k\}$. $N(0) = 0$, $N(\infty) = ab$.
- For each $i > 0$, $L_i = \{N(i-1)+1, N(i-1)+2, \ldots, N(i)\}$.

Step 1: Scan the rightmost column of M from bottom to top, replace each i as it is encountered by the largest unused value in L_i.

Step j: For each j from $b-1$ down to 1, scan column j in the reverse order determined by the values in column $j+1$ of $std(M)$, replace each i as it is encountered by the largest unused value in L_i.

$$M = \begin{pmatrix} 1 & 1 & 3 & 3 & 5 \\ 1 & 2 & 2 & 2 & 4 \\ 2 & 2 & 3 & 3 & 3 \end{pmatrix} \quad std(M) = \begin{pmatrix} 1 & 3 & 10 & 12 & 15 \\ 2 & 5 & 7 & 8 & 14 \\ 4 & 6 & 9 & 11 & 13 \end{pmatrix},$$

$sort(M) = 111222223333345.$

Plethysm and QSYM

Thm.(Loehr-Warrington 2012) For any two partitions μ, ν

$$s_\mu[s_\nu(X)] = \sum_{A \in M(\mu,\nu)} wt(A) = \sum_{A \in S_{a,b}(\mu,\nu)} F_{Des(rw(A)^{-1})}(X)$$

where $S_{a,b}(\mu,\nu)$ is the set of standard $a \times b$-matrices which respect the conditions determined by $SSYT_{\mathcal{W}(\nu)}(\mu)$ and $Des(rw(A)^{-1})$ is the descent set of the reading word permutation associated to A.

Bijective Proof. Three things to check:

1. The map S taking $M \in M_{a,b}$ to $(std(M), sort(M))$ is invertible.
2. The sequence $sort(M)$ is compatible with $Des(rw(std(M))^{-1})$.
3. The map std on $M_{a,b}$ maps $M(\mu,\nu)$ into $S_{a,b}(\mu,\nu)$.

Plethysm and QSYM

Thm. (Loehr-Warrington 2012) For any two partitions μ, ν

$$s_\mu[s_\nu(X)] = \sum_{A \in S_{a,b}(\mu,\nu)} F_{Des(rw(A)^{-1})}.$$

Example.

$$s_{(2)}[s_{(3)}] = F_{[3,3]} + F_{[2,2,2]} + F_{[2,4]} + F_{[2,3,1]} + F_{[4,2]} + F_{[6]} + F_{[3,2,1]} + F_{[1,2,3]} + F_{[1,4,1]} + F_{[1,3,2]}$$

A	$\begin{bmatrix}1\,2\,3\\4\,5\,6\end{bmatrix}$	$\begin{bmatrix}1\,2\,4\\3\,5\,6\end{bmatrix}$	$\begin{bmatrix}1\,2\,5\\3\,4\,6\end{bmatrix}$	$\begin{bmatrix}1\,2\,6\\3\,4\,5\end{bmatrix}$	$\begin{bmatrix}1\,3\,4\\2\,5\,6\end{bmatrix}$
rw(A)	142536	132546	132456	134265	123546
α	33	222	24	231	42
A	$\begin{bmatrix}1\,3\,5\\2\,4\,6\end{bmatrix}$	$\begin{bmatrix}1\,3\,6\\2\,4\,5\end{bmatrix}$	$\begin{bmatrix}1\,4\,5\\2\,3\,6\end{bmatrix}$	$\begin{bmatrix}1\,4\,6\\2\,3\,5\end{bmatrix}$	$\begin{bmatrix}1\,5\,6\\2\,3\,4\end{bmatrix}$
rw(A)	123456	124365	214356	213465	213564
α	6	321	123	141	132

Open Problem. Expand $s_\mu[s_\nu(X)]$ in Schur basis and relate back to "P vs NP".

Schur positive expansions

Recently Established Methods.

1. Use Dual Equivalence Graphs.
(Assaf '08-'13, Roberts '13-'14)
2. Flip F_α to s_α.
(Egge-Loehr-Warrington 2010)
3. Find a quasi-Schur expansion: $s_\lambda = \sum_{\alpha : sort(\alpha)=\lambda} \mathcal{S}_\alpha$.
(Haglund-Luoto-Mason-vanWilligenburg 2011, book 2013)

High level goals

1. Develop intuition for some of the tools in algebraic combinatorics.
2. Build up vocabulary to introduce some important open problems and approaches to attack them.
3. Inspire you to learn more about quasisymmetric functions and find more applications.

¡Muchas Gracias!