INDECOMPOSABLE MODULES FOR THE DUAL IMMACULATE BASIS OF QUASI-SYMMETRIC FUNCTIONS

CHRIS BERG[2], NANTEL BERGERON[1,3], FRANCO SALIOLA[2], LUIS SERRANO[2], AND MIKE ZABROCKI[1,3]

Abstract. We construct indecomposable modules for the 0-Hecke algebra whose characteristics are the dual immaculate basis of the quasi-symmetric functions.

1. Introduction

The algebra of symmetric functions Sym has an important basis formed by Schur functions, which appear throughout mathematics. For example, as the representatives for the Schubert classes in the cohomology of the Grassmannian, as the characters for the irreducible representations of the symmetric group and the general linear group, or as an orthonormal basis for the space of symmetric functions, to name a few. The algebra NSym of noncommutative symmetric functions projects under the forgetful map onto Sym, which injects into the algebra QSym of quasi-symmetric functions. NSym and QSym are dual Hopf algebras.

In [BBSSZ], the authors developed a basis for NSym, which satisfied many of the combinatorial properties of Schur functions. This basis, called the *immaculate basis* $\{\mathfrak{S}_\alpha\}$, projects onto Schur functions under the forgetful map. When indexed by a partition, the corresponding projection of the immaculate function is precisely the Schur function of the given partition.

The dual basis $\{\mathfrak{S}^*_\alpha\}$ is a basis for QSym. The main goal of this paper is to express the dual immaculate functions as characters of a representation, in the same way that Schur functions are the characters of the irreducible representations of the symmetric group. We achieve this in Theorem 3.5, where we realize them as the characteristic of certain indecomposable representations of the *0-Hecke algebra*.

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2. Prerequisites

2.1. The symmetric group. The symmetric group S_n is the group generated by the set of $\{s_1, s_2, \ldots, s_{n-1}\}$ satisfying the following relations:

$$\begin{aligned} s_i^2 &= 1; \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}; \\ s_i s_j &= s_j s_i \text{ if } |i-j| > 1. \end{aligned}$$

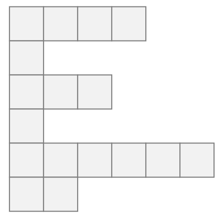
2.2. Compositions and combinatorics. A *partition* of a non-negative integer n is a tuple $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m]$ of positive integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ which sum to n. If λ is a partition of n, one writes $\lambda \vdash n$. (When needed, we will consider partitions with zeroes at the end, but they are equivalent to the underlying partition made of positive numbers.) Partitions are of particular importance in algebraic combinatorics, as they index a basis for the symmetric functions of degree n, Sym_n, and the character ring for the representations of the symmetric group S_n, among others. These concepts are intimately connected; we assume the reader is well versed in this area (see for instance [Sagan] for background details).

A *composition* of a non-negative integer n is a tuple $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m]$ of positive integers which sum to n. If α is a composition of n, one writes $\alpha \models n$. The entries α_i of the composition are referred to as the parts of the composition. The size of the composition is the sum of the parts and will be denoted $|\alpha|$. The length of the composition is the number of parts and will be denoted $\ell(\alpha)$. Note that $|\alpha| = n$ and $\ell(\alpha) = m$.

Compositions of n are in bijection with subsets of $\{1, 2, \ldots, n-1\}$. We will follow the convention of identifying $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m]$ with the subset $\mathcal{S}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{m-1}\}$.

If α and β are both compositions of n, say that $\alpha \leq \beta$ in refinement order if $\mathcal{S}(\beta) \subseteq \mathcal{S}(\alpha)$. For instance, $[1,1,2,1,3,2,1,4,2] \leq [4,4,2,7]$, since $\mathcal{S}([1,1,2,1,3,2,1,4,2]) = \{1,2,4,5,8,10,11,15\}$ and $\mathcal{S}([4,4,2,7]) = \{4,8,10\}$.

In this presentation, compositions will be represented as diagrams of left adjusted rows of cells. We will also use the matrix convention ('English' notation) that the first row of the diagram is at the top and the last row is at the bottom. For example, the composition $[4, 1, 3, 1, 6, 2]$ is represented as



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2.3. **Symmetric functions.** We let Sym denote the ring of symmetric functions. As an algebra, Sym is the ring over $\mathbb{Q}$ freely generated by commutative elements $\{h_1, h_2, \dots\}$. Sym has a grading, defined by giving h_i degree i and extending multiplicatively. A natural basis for the degree n component of Sym are the complete homogeneous symmetric functions of degree n, $\{h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_m} : \lambda \vdash n\}$. Sym can be realized as the ring of invariants of the ring of power series of bounded degree $\mathbb{Q}[\![x_1, x_2, \dots]\!]$ in commuting variables $\{x_1, x_2, \dots\}$. Under this identification, h_i denotes the sum of all monomials in the x variables of degree i.

2.4. **Non-commutative symmetric functions.** NSym is a non-commutative analogue of Sym, the algebra of symmetric functions, that arises by considering an algebra with one non-commutative generator at each positive degree. In addition to the relationship with the symmetric functions, this algebra has links to Solomon's descent algebra in type A [MR], the algebra of quasi-symmetric functions [MR], and representation theory of the type A Hecke algebra at $q = 0$ [KT]. It is an example of a combinatorial Hopf algebra [ABS]. While we will follow the foundational results and definitions from references such as [GKLLRT, MR], we have chosen to use notation here which is suggestive of analogous results in Sym.

We consider NSym as the algebra with generators $\{H_1, H_2, \dots\}$ and no relations. Each generator H_i is defined to be of degree i, giving NSym the structure of a graded algebra. We let NSym_n denote the graded component of NSym of degree n. A basis for NSym_n are the *complete homogeneous functions* $\{H_\alpha := H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_m}\}_{\alpha \models n}$ indexed by compositions of n.

2.5. **Immaculate tableaux.**

Definition 2.1. Let α and β be compositions. An *immaculate tableau* of shape α and content β is a labelling of the boxes of the diagram of α by positive integers in such a way that:

(1) the number of boxes labelled by i is β_i;
(2) the sequence of entries in each row, from left to right, is weakly increasing;
(3) the sequence of entries in the *first* column, from top to bottom, is increasing.

An immaculate tableau is said to be *standard* if it has content $1^{|\alpha|}$.

Let $K_{\alpha,\beta}$ denote the number of immaculate tableaux of shape α and content β.

We re-iterate that aside from the first column, there is no relation on the other columns of an immaculate tableau.

Example 2.2. *The five immaculate tableau of shape* $[4, 2, 3]$ *and content* $[3, 1, 2, 3]$*:*

1	1	1	3
2	3		
4	4	4	

1	1	1	3
2	4		
3	4	4	

1	1	1	4
2	3		
3	4	4	

1	1	1	4
2	4		
3	3	4	

1	1	1	2
3	3		
4	4	4	

Definition 2.3. We say that a standard immaculate tableau T has a descent in position i if $i + 1$ is in a row strictly below the row containing i. The *descent composition*, denoted $D(T)$, is the composition corresponding to the set of descents in T.

Example 2.4. *The standard immaculate tableau of shape* $[6, 5, 7]$:

$T =$

1	2	4	5	10	11	
3	6	7	8	9		
12	13	14	15	16	17	18

has descents in positions $\{2, 5, 11\}$*. The descent composition of* T *is then* $D(T) = [2, 3, 6, 7]$.

2.6. **The immaculate basis of NSym.** The immaculate basis of NSym was introduced in [BBSSZ]. It shares many properties with the Schur basis of Sym. We define[1] the immaculate basis $\{\mathfrak{S}_\alpha\}_\alpha$ as the unique elements of NSym satisfying:

$$H_\beta = \sum_\alpha K_{\alpha,\beta} \mathfrak{S}_\alpha.$$

Example 2.5. *Continuing from Example 2.2, we see that*

$$H_{3123} = \cdots + 5\mathfrak{S}_{423} + \cdots .$$

We will not attempt to summarize everything that is known about this basis, but instead refer the reader to [BBSSZ] and [BBSSZ2].

3. Modules for the dual immaculate basis

In this section we will construct indecomposable modules for the 0-Hecke algebra whose characteristic is a dual immaculate quasi-symmetric function.

[1]This is not the original definition, but is equivalent by Proposition 3.16 in [BBSSZ].

3.1. **Quasi-symmetric functions.** The algebra of quasi-symmetric functions, QSym, was introduced in [Ges] (see also subsequent references such as [GR, Sta84]). The graded component QSym_n is indexed by compositions of n. The algebra is most readily realized as a subalgebra of the ring of power series of bounded degree $\mathbb{Q}[\![x_1, x_2, \dots]\!]$, and the monomial quasi-symmetric function indexed by a composition α is defined as

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_m} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_m}^{\alpha_m}. \tag{1}$$

The algebra of quasi-symmetric functions, QSym, can be defined as the linear span of the monomial quasi-symmetric functions. These, in fact, form a basis of QSym, and their multiplication is inherited from $\mathbb{Q}[\![x_1, x_2, \dots]\!]$. We view Sym as a subalgebra of QSym. In fact, the quasi-symmetric monomial functions refine the usual monomial symmetric functions $m_\lambda \in \mathsf{Sym}$:

$$m_\lambda = \sum_{\mathrm{sort}(\alpha)=\lambda} M_\alpha,$$

where $\mathrm{sort}(\alpha)$ denotes the partition obtained by organizing the parts of α from the largest to the smallest.

The fundamental quasi-symmetric function, denoted F_α, is defined by its expansion in the monomial quasi-symmetric basis:

$$F_\alpha = \sum_{\beta \le \alpha} M_\beta.$$

The algebras QSym and NSym form dual graded Hopf algebras. In this context, the monomial basis of QSym is dual to the complete homogeneous basis of NSym. Duality can be expressed by the means of an inner product, for which $\langle H_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$.

In [BBSSZ], we studied the dual basis to the immaculate functions of NSym, denoted $\mathfrak{S}^*_\beta$ and indexed by compositions. They are the basis of QSym defined by $\langle \mathfrak{S}_\alpha, \mathfrak{S}^*_\beta \rangle = \delta_{\alpha,\beta}$. In [BBSSZ, Proposition 3.37], we showed that the dual immaculate functions have the following positive expansion into the fundamental basis:

Proposition 3.1. *The dual immaculate functions $\mathfrak{S}^*_\alpha$ are fundamental positive. Specifically they expand as*

$$\mathfrak{S}^*_\alpha = \sum_T F_{D(T)},$$

a sum over all standard immaculate tableaux of shape α.

3.2. **Finite dimensional representation theory of $H_n(0)$.** We will outline the study of the finite dimensional representations of the 0-Hecke algebra and its relationship to QSym. We begin by defining the 0-Hecke algebra. We refer the reader to [Th2, Section 5] for the relationship between the generic Hecke algebra and the 0-Hecke algebra and their connections to representation theory.

Definition 3.2. The Hecke algebra $H_n(0)$ is generated by the elements $\pi_1, \pi_2, \ldots \pi_{n-1}$ subject to relations:

$$\begin{aligned} \pi_i^2 &= \pi_i; \\ \pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1}; \\ \pi_i \pi_j &= \pi_j \pi_i \text{ if } |i-j| > 1. \end{aligned}$$

A basis of $H_n(0)$ is given by the elements $\{\pi_\sigma : \sigma \in S_n\}$, where $\pi_\sigma = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ if $\sigma = s_{i_1}s_{i_2}\cdots s_{i_m}$.

We let $G_0(H_n(0))$ denote the Grothendieck group of finite dimensional representations of $H_n(0)$. As a vector space, $G_0(H_n(0))$ is spanned by the finite dimensional representations of $H_n(0)$, with the relation on isomorphism classes $[B] = [A] + [C]$ whenever there is a short exact sequence of $H_n(0)$-representations $0 \to A \to B \to C \to 0$. We let

$$\mathcal{G} = \bigoplus_{n \geq 0} G_0(H_n(0)).$$

The irreducible representations of $H_n(0)$ are indexed by compositions. The irreducible representation corresponding to the composition α is denoted L_α. The collection $\{[L_\alpha]\}$ forms a basis for $\mathcal{G}$. As shown in Norton [N], each irreducible representation is one dimensional, spanned by a non-zero vector $v_\alpha \in L_\alpha$, and is determined by the action of the generators on v_α:

$$\pi_i v_\alpha = \begin{cases} 0 & \text{if } i \in \mathcal{S}(\alpha); \\ v_\alpha & \text{otherwise,} \end{cases} \tag{2}$$

where $\mathcal{S}(\alpha)$ denotes the subset of $[1 \ldots n-1]$ corresponding to the composition α. The tensor product $H_n(0) \otimes H_m(0)$ is naturally embedded as a subalgebra of $H_{n+m}(0)$. Under this identification, one can endow $\mathcal{G}$ with a ring structure; for $[N] \in G_0(H_n(0))$ and $[M] \in G_0(H_m(0))$, let

$$[N][M] := [Ind_{H_n(0)\otimes H_m(0)}^{H_{n+m}(0)} N \otimes M]$$

where induction is defined in the usual manner.

There is an important linear map $\mathcal{F} : \mathcal{G} \to \mathsf{QSym}$ defined by $\mathcal{F}([L_\alpha]) = F_\alpha$. For a module M, $\mathcal{F}([M])$ is called the *characteristic of M*.

Theorem 3.3 (Duchamp, Krob, Leclerc, Thibon [DKLT])**.** *The quasi-symmetric functions and the Grothendieck group of finite dimensional representations of $H_n(0)$ are isomorphic as rings. The map $\mathcal{F}$ is an isomorphism between $\mathcal{G}$ and* QSym*.*

Remark 3.4. The map $\mathcal{F}$ is actually an isomorphism of graded Hopf algebras. We will not make use of the coalgebra structure.

3.3. **A representation on $\mathcal{Y}$-words.** We start by defining the analogue of a permutation module for $H_n(0)$. For a composition $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m] \models n$, we let $\mathcal{M}_\alpha$ denote the vector space spanned by words of length n on m letters with content α (so that j appears α_j times in each word). The action of $H_n(0)$ on a word $w = w_1 w_2 \cdots w_n$ is defined on generators as:

$$\pi_i w = \begin{cases} w & \text{if } w_i \geq w_{i+1}; \\ s_i(w) & \text{if } w_i < w_{i+1}; \end{cases} \tag{3}$$

where $s_i(w) = w_1 w_2 \cdots w_{i+1} w_i \cdots w_n$. This is isomorphic to the representation:

$$Ind_{H_\alpha(0)}^{H_n(0)} \left(L_{\alpha_1} \otimes L_{\alpha_2} \otimes \cdots \otimes L_{\alpha_m} \right),$$

where L_k is the one-dimensional representation indexed by the composition $[k]$ and $H_\alpha(0) := H_{\alpha_1}(0) \otimes H_{\alpha_2}(0) \otimes \cdots \otimes H_{\alpha_m}(0)$. This can be seen by associating the element $\pi_v \otimes_{H_\alpha(0)} L_{\alpha_1} \otimes L_{\alpha_2} \otimes \cdots \otimes L_{\alpha_m}$ where v is the minimal length left coset representative of $S_n / S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_m}$ with the element $\pi_v(1^{\alpha_1} 2^{\alpha_2} \cdots k^{\alpha_k})$.

We call a word a *$\mathcal{Y}$-word* if the first instance of j appears before the first instance of $j+1$ for every j. We let $\mathcal{N}_\alpha$ denote the subspace of $\mathcal{M}_\alpha$ consisting of all words that are not $\mathcal{Y}$-words. The action of $H_n(0)$ on $\mathcal{M}_\alpha$ will never move a $j+1$ to the right of a j. This implies that $\mathcal{N}_\alpha$ is a submodule of $\mathcal{M}_\alpha$. The object of our interest is the quotient module $\mathcal{V}_\alpha := \mathcal{M}_\alpha / \mathcal{N}_\alpha$. We now state our main result.

Theorem 3.5. *The characteristic of $\mathcal{V}_\alpha$ is the dual immaculate function indexed by α, i.e. $\mathcal{F}([\mathcal{V}_\alpha]) = \mathfrak{S}^*_\alpha$.*

Before we prove this we will associate words to standard immaculate tableaux and give an equivalent description of the 0-Hecke algebra on standard immaculate tableau. To a $\mathcal{Y}$-word w, we associate the unique standard immaculate tableau $\mathcal{T}(w)$ which has a j in row w_j.

Example 3.6. *Let $w = 112322231$ be the $\mathcal{Y}$-word of content $[3, 4, 2]$. Then $\mathcal{T}(w)$ is the standard immaculate tableau:*

1	2	9	
3	5	6	7
4	8		

Remark 3.7. $\mathcal{T}$ yields a bijection between standard immaculate tableau and $\mathcal{Y}$-words.

Remark 3.8. In the case of the symmetric group, the irreducible representation corresponding to the partition λ has a basis indexed by standard tableaux. Under the same map $\mathcal{T}$, standard Young tableaux are in bijection with Yamanouchi words (words for which every prefix contains at least as many j as $j+1$ for every j). In this sense, $\mathcal{Y}$-words are a natural analogue to Yamanouchi words in our setting. The Specht modules that give rise to the indecomposable module of the symmetric group are built as a quotient of $\mathcal{M}_\lambda$. Under the Frobenius map, these modules are associated to Schur functions.

We may now describe the action of $H_n(0)$ on $\mathcal{V}_\alpha$, identifying the set of standard immaculate tableaux as the basis. Specifically, for a tableau T and a generator π_i, we let:

$$\pi_i(T) = \begin{cases} 0 & \text{if } i \text{ and } i{+}1 \text{ are in the first column of } T \\ T & \text{if } i \text{ is in a row weakly below the row containing } i{+}1 \\ s_i(T) & \text{otherwise;} \end{cases} \tag{4}$$

where $s_i(T)$ is the tableau that differs from T by swapping the letters i and $i+1$.

Example 3.9. *Continuing from Example 3.6, we see that $\pi_1, \pi_4, \pi_5, \pi_6, \pi_8$ send T to itself, π_3 sends T to 0 and π_2, π_7 send T to the following tableaux:*

$$\pi_2(T) = \begin{array}{|c|c|c|c} \hline 1 & 3 & 9 & \multicolumn{1}{|c}{} \\ \hline 2 & 5 & 6 & \multicolumn{1}{|c|}{7} \\ \hline 4 & 8 & \multicolumn{1}{|c}{} & \\ \cline{1-2} \end{array} \qquad \pi_7(T) = \begin{array}{|c|c|c|c} \hline 1 & 2 & 9 & \multicolumn{1}{|c}{} \\ \hline 3 & 5 & 6 & \multicolumn{1}{|c|}{8} \\ \hline 4 & 7 & \multicolumn{1}{|c}{} & \\ \cline{1-2} \end{array}$$

An example of the full action of π_i on tableaux representing the basis elements of the module $\mathcal{V}_{(2,2,3)}$ is given in Figure 3.3. If we order the tableaux so that $S \prec T$ if there exists a permutation σ such that $\pi_\sigma(T) = S$ then this figure shows that order is not a total order on tableaux but that it can be extended to a total order arbitrarily. We will use this total order in the following proof of Theorem 3.5.

We are now ready to prove Theorem 3.5, which states that the characteristic of $\mathcal{V}_\alpha$ is $\mathfrak{S}^*_\alpha$.

Proof of Theorem 3.5. We construct a filtration of the module $\mathcal{V}_\alpha$ whose successive quotients are irreducible representations. Now, define $\mathcal{M}_T$ to be the linear span of all standard immaculate tableaux that are less than or equal to T. From the definition of the order and the fact that the π_i are not invertible, we see that $\mathcal{M}_T$ is a module. Ordering the standard immaculate tableaux of shape α as $T_1, T_2, \ldots, T_m$, then we have a filtration of $\mathcal{V}_\alpha$:

$$0 \subset \mathcal{M}_{T_1} \subset \mathcal{M}_{T_2} \subset \cdots \subset \mathcal{M}_{T_m} = \mathcal{V}_\alpha.$$

The successive quotient modules $\mathcal{M}_{T_j}/\mathcal{M}_{T_{j-1}}$ are one dimensional, spanned by T_j; to determine which irreducible this is, it suffices to compute the action of the generators. From the description of $\mathcal{V}_\alpha$ above, we see that

$$\pi_i(T_j) = \begin{cases} 0 & \text{if } i \in \mathcal{S}(D(T_j)) \\ T_j & \text{otherwise.} \end{cases} \tag{5}$$

This is the representation $[L_{D(T_j)}]$, whose characteristic is $F_{D(T_j)}$. Therefore $\mathcal{F}([\mathcal{V}_\alpha]) = \mathfrak{S}^*_\alpha$ by Proposition 3.1. □

We aim to prove that the modules we have constructed are indecomposable. We let $\hat{S}_\alpha$ denote the super-standard tableau of shape α, namely, the unique standard immaculate tableau with the first α_1 letters in the first row, the next α_2 letters in the second row, etc. We will first need a few lemmas.

Lemma 3.10. *The module $\mathcal{V}_\alpha$ is cyclicly generated by $\hat{S}_\alpha$.*

Proof. The module $\mathcal{M}_\alpha$ is cyclically generated by $1^{\alpha_1}2^{\alpha_2}\cdots k^{\alpha_k} = \mathcal{T}^{-1}(\hat{S}_\alpha)$, which can be seen since every basis element of $\mathcal{M}_\alpha$ comes from an application of the anti-sorting operators π_i on $1^{\alpha_1}2^{\alpha_2}\cdots k^{\alpha_k}$.

$\mathcal{V}_\alpha$ is a quotient of $\mathcal{M}_\alpha$, and hence cyclicly generated by the same element. □

Lemma 3.11. *If P is a standard immaculate tableau of shape α such that $\pi_i(P) = P$ for all $i \in \{1, 2, \cdots, n\} \setminus \mathcal{S}(\alpha)$ then $P = \hat{S}_\alpha$. In particular, if $P \neq \hat{S}_\alpha$ then there exists an i such that $\pi_i(\hat{S}_\alpha) = \hat{S}_\alpha$ but $\pi_i(P) \neq P$.*

Proof. If $\pi_i(P) = P$, then i must be in the cell to the left of $i+1$ or in a row below $i+1$. The fact that $\pi_i(P) = P$ for all $i \in \{1, 2, \ldots, \alpha_1 - 1\}$ implies that the first row of P agrees with $\hat{S}_\alpha$. In a similar manner, we see that the second rows must agree. Continuing in this manner, we conclude that $P = \hat{S}_\alpha$. □

Theorem 3.12. *For every $\alpha \models n$, $\mathcal{V}_\alpha$ is an indecomposable representation of $H_n(0)$.*

Proof. We let f be an idempotent module morphism from $\mathcal{V}_\alpha$ to itself. If we can prove f is either the zero morphism or the identity, then $\mathcal{V}_\alpha$ is indecomposable [Ja, Proposition 3.1].

Suppose $f(\hat{S}_\alpha) = \sum_T a_T T$. By Lemma 3.11, for any $P \neq \hat{S}_\alpha$, there exists an i such that $\pi_i(\hat{S}_\alpha) = \hat{S}_\alpha$ but $\pi_i(P) \neq P$. Since f is a module map,

$$\sum_T a_T T = f(\hat{S}_\alpha) = f(\pi_i \hat{S}_\alpha) = \pi_i f(\hat{S}_\alpha) = \sum_T a_T \pi_i T. \tag{6}$$

The coefficient of P on the right-hand side of Equation (6) is zero (if there was a T such that $\pi_i T = P$ then $\pi_i T = \pi_i^2 T = \pi_i P \neq P$, a contradiction). Therefore $a_P = 0$ for all $P \neq \hat{S}_\alpha$, so $f(\hat{S}_\alpha) = \hat{S}_\alpha$, or $f(\hat{S}_\alpha) = 0$. Since $\mathcal{V}_\alpha$ is cyclicly generated by $\hat{S}_\alpha$, this implies that either f is the identity morphism or the zero morphism. □

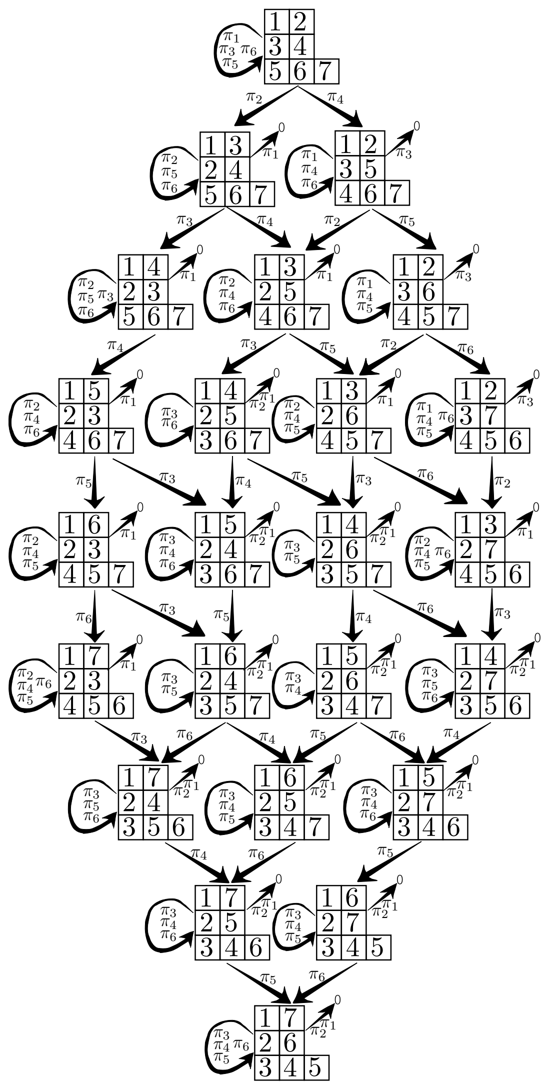

FIGURE 1. A diagram representing the action of the generators π_i of $H_n(0)$ given in Equation (4) on the basis elements of the module $\mathcal{V}_{(2,2,3)}$.

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(1) Fields Institute, Toronto, ON, Canada

(2) Université du Québec à Montréal, Montréal, QC, Canada

(3) York University, Toronto, ON, Canada