

# THE PEAK ALGEBRA AND THE HECKE-CLIFFORD ALGEBRAS AT $q = 0$

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ABSTRACT. Using the formalism of noncommutative symmetric functions, we derive the basic theory of the peak algebra of symmetric groups and of its graded Hopf dual. Our main result is to provide a representation theoretical interpretation of the peak algebra and its graded dual as Grothendieck rings of the tower of Hecke-Clifford algebras at  $q = 0$ .

## 1. INTRODUCTION

Studies on the combinatorics of descents in permutations led to the discovery of a pair,  $(QSym, \mathbf{Sym})$ , of mutually dual graded Hopf algebras [9, 15]. Here,  $QSym$  is the graded Hopf algebra of quasi-symmetric functions, and its graded dual,  $\mathbf{Sym}$ , is the graded Hopf algebra of noncommutative symmetric functions. Recent investigations on the combinatorics of peaks in permutations resulted in the discovery of an interesting new pair,  $(\mathbf{Peak}, \mathbf{Peak}^*)$ , of graded Hopf algebras. The first one,  $\mathbf{Peak}$ , originally due to Stembridge [21], is a subalgebra of  $QSym$ . As described in [3], its graded dual,  $\mathbf{Peak}^*$ , can therefore be identified as a homomorphic image of  $\mathbf{Sym}$ . We shall see in the following that the existence of  $\mathbf{Peak}^*$  as well as many of its basic properties were already implicit in [12].

It is known that  $\mathbf{Peak}$  can also be obtained as a quotient of  $QSym$ , in which case  $\mathbf{Peak}^*$  is realized as a subalgebra of  $\mathbf{Sym}$ . On the other hand, each homogeneous component  $\mathbf{Sym}_n$  of  $\mathbf{Sym}$  is endowed with another multiplication, the internal product  $*$ , such that the resulting algebra is anti-isomorphic to Solomon's descent algebra of the symmetric group  $\mathfrak{S}_n$ . At this stage, a natural question arises. Is  $\mathbf{Peak}_n^*$  stable under this operation? As shown in [16], the answer is yes (it is even a left ideal of  $\mathbf{Sym}_n$ ), and the corresponding right ideal of the descent algebra is spanned by the sums of permutations having a given peak set. Recent developments [1, 2, 5, 19] unveil many interesting properties and generalizations of  $\mathbf{Peak}$  and  $\mathbf{Peak}^*$ . Most notably, we find in [2] that  $\mathbf{Peak}$  is the terminal object in the category of combinatorial Hopf algebras satisfying generalized Dehn-Somerville relations. This reveals some of the significance of  $\mathbf{Peak}$  and  $\mathbf{Peak}^*$ . Our main result demonstrates yet another facet of the importance of these graded Hopf algebras.

We shall start our presentation by showing that many of the basic results in the literature related to  $\mathbf{Peak}$  and  $\mathbf{Peak}^*$  can be recovered in a very elegant and straightforward way by relying upon the techniques developed in [12]. This will be covered in Sections 2,3 and 4.

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It is known that the dual pair of Hopf algebras  $(QSym, \mathbf{Sym})$  describes the representation theory of the 0-Hecke algebras of type  $A$  [13]. More precisely,  $QSym$  and  $\mathbf{Sym}$  are respectively isomorphic to the direct sums of the Grothendieck groups  $G_0(H_n(0))$  and  $K_0(H_n(0))$ . We provide here a similar interpretation for the pair  $(\mathbf{Peak}, \mathbf{Peak}^*)$ . This is done by replacing the Hecke algebras with the so-called Hecke-Clifford algebras, discovered by G. Olshanski [17]. This new result is presented in Section 5.

We will assume that the reader is familiar with the notation of [9, 12].

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## 2. THE $(1 - q)$ -TRANSFORM AT $q = -1$

The main motivation for Stembridge’s theory of enriched  $P$ -partitions, which led him to the quasi-symmetric peak algebra [21], was the study of the quasi-symmetric expansions of Schur’s  $Q$ -functions [20, 14]. As is well known, these symmetric functions correspond to the Hall-Littlewood functions with parameter  $q = -1$ . The peak algebra is therefore directly related to what we will call the “ $(1 - q)$ -transform” at  $q = -1$ .

In the classical case, the  $(1 - q)$ -transform  $\theta_q$  is the ring endomorphism of  $Sym$  defined on the power sums by  $\theta_q(p_n) = (1 - q^n)p_n$ . In the  $\lambda$ -ring notation, which is particularly convenient to deal with such transformations, it reads  $f(X) \mapsto f((1 - q)X)$ . One has to pay attention to the abuse of notation in using the same minus sign for the  $\lambda$ -ring and for scalars, though these operations are quite different. That is,  $\theta_{-1}$  maps  $p_n$  to  $2p_n$  if  $n$  is odd, and to 0 otherwise. Thus,  $\theta_{-1}(f(X)) = f((1 - q)X)_{q=-1}$  is not the same as  $f((1 + 1)X) = f(2X)$ .

The main results of [12] are concerned with the extension of the  $(1 - q)$ -transform to noncommutative symmetric functions. More precisely, consider the abelianization map  $\chi: \mathbf{Sym} \rightarrow Sym$  which sends the noncommutative alphabet  $A$  to the commutative alphabet  $X$ . We are interested in defining a  $(1 - q)$ -transform on  $\mathbf{Sym}$  which commutes with  $\chi$ . A consistent definition of  $\theta_q(F) = F((1 - q)A)$  is proposed, and its fundamental properties are obtained. We briefly recall here the necessary steps. One first defines the complete symmetric functions  $S_n((1 - q)A)$  via their generating series [12, Def. 5.1]

$$(1) \quad \sigma_t((1 - q)A) := \sum_{n \geq 0} t^n S_n((1 - q)A) = \left( \sum_{n \geq 0} (-qt)^n \Lambda_n(A) \right) \left( \sum_{n \geq 0} t^n S_n(A) \right),$$

and then  $\theta_q$  is defined as the ring homomorphism such that  $\theta_q(S_n) = S_n((1 - q)A)$ . Specializing [12, Thm. 4.17] to our case, we obtain

$$(2) \quad F((1 - q)A) = F(A) * \sigma_1((1 - q)A),$$

where  $*$  is the internal product of  $\mathbf{Sym}$ .

The most important property of  $\theta_q$  is its diagonalization [12, Thm. 5.14]: there is a unique family of Lie idempotents  $\pi_n(q)$  (i.e., elements in the primitive Lie algebra

such that  $\chi(\pi_n(q)) = \frac{1}{n}p_n$  with the property

$$(3) \quad \theta_q(\pi_n(q)) = (1 - q^n)\pi_n(q).$$

Moreover,  $\theta_q$  is semi-simple, and its eigenvalues in  $\mathbf{Sym}_n$  are  $p_\lambda(1 - q) = \prod_i (1 - q^{\lambda_i})$  where  $\lambda$  runs over the partitions of  $n$ . The projectors on the corresponding eigenspaces are the maps  $F \mapsto F * \pi^I(q)$ , where for a composition  $I = (i_1, i_2, \dots, i_r)$ , we let  $\pi^I(q) = \pi_{i_1}(q)\pi_{i_2}(q) \cdots \pi_{i_r}(q)$  [12, Sec. 3.4].

Another result [12, Sec. 5.6.4], which is just a translation of an important formula due to Blessenohl and Laue [6], gives  $\theta_q(R_I)$  in closed form for any ribbon  $R_I$ . To be more in line with the current literature, we digress slightly from the notation of [9]. Let  $[i, j] = \{i, i + 1, i + 2, \dots, j\}$ . Given a composition  $I = (i_1, i_2, \dots, i_r)$  of  $n$ , let  $\text{Des}(I) = \{i_1, i_1 + i_2, \dots, i_1 + \dots + i_{r-1}\} \subseteq [1, n - 1]$  denote the *descent set* of  $I$ . We let  $A \Delta B = (A - B) \cup (B - A)$  be the symmetric difference of two sets. Given  $A = \{a_1, a_2, \dots, a_{r-1}\} \subseteq [1, n - 1]$  we let  $A + 1 = \{a_1 + 1, a_2 + 1, \dots, a_{r-1} + 1\} \subseteq [2, n]$ . For a composition  $J$  of  $n$ , one defines  $HP(J) = \{a \in \text{Des}(J) \mid a \neq 1, a - 1 \notin \text{Des}(J)\} \subseteq [2, n - 1]$ , and  $hl(J) = |HP(J)| + 1$ . One usually refers to  $HP(J)$  as the *peak set* of  $J$ . We are now in a position to give the formula for  $\theta_q(R_I)$  [12, Lem. 5.38 and Prop. 5.41]:

$$(4) \quad R_I((1 - q)A) = \sum_{HP(J) \subseteq \text{Des}(I) \Delta (\text{Des}(I) + 1)} (1 - q)^{hl(J)} (-q)^{b(I, J)} R_J(A),$$

where  $b(I, J)$  is some explicit integer, but is not of any use when  $q = -1$ .

Setting  $q = -1$  in the formulas above leads us immediately to the peak classes in  $\mathbf{Sym}$ . We say that a set  $P \subseteq [2, n - 1]$  is a *peak set* when  $a \in P \implies a - 1 \notin P$ . For a peak set  $P$  let

$$(5) \quad \Pi_P = \sum_{HP(I) = P} R_I.$$

At  $q = -1$ , Eq. (4) now reads as

$$(6) \quad \theta_{-1}(R_I) = \sum_{P \subseteq \text{Des}(I) \Delta (\text{Des}(I) + 1)} 2^{|P| + 1} \Pi_P,$$

which is [19, Prop. 5.5] or [1, Prop. 5.8].

Let us denote for short  $\theta_{-1}$  by a tilde,  $\tilde{F} := \theta_{-1}(F)$ , and let  $\widetilde{\mathbf{Sym}}$  be its image. Since by definition

$$(7) \quad \widetilde{\mathbf{Sym}} = \{F((1 - q)A)_{q=-1}\}$$

it is immediate that  $\widetilde{\mathbf{Sym}}$  is a graded Hopf subalgebra of  $\mathbf{Sym}$ . Indeed,  $F \mapsto F((1 - q)A)$  is an algebra morphism, and also a coalgebra morphism, since [12, Sec. 5.1]

$$(8) \quad \Delta S_n((1 - q)A) = \sum_{i+j=n} S_i((1 - q)A) \otimes S_j((1 - q)A)$$

for all values of  $q$ . Also, it is a left ideal for the internal product, since by Eq. (2)

$$(9) \quad \widetilde{\mathbf{Sym}} = \mathbf{Sym}(A) * \sigma_1((1 - q)A)_{q=-1}.$$

We already know that  $\widetilde{\mathbf{Sym}}$  is contained in the subspace  $\mathcal{P}$  of  $\mathbf{Sym}$  spanned by the peak classes. The dimension of the homogeneous component  $\mathcal{P}_n$  of  $\mathcal{P}$  is easily seen to be equal to the Fibonacci number  $f_n$  (with the convention  $f_0 = f_1 = f_2 = 1$ ,  $f_{n+2} = f_{n+1} + f_n$ ). Indeed, the set of compositions of  $n$  having a given peak set  $P$  has a unique minimal and maximal element. The minimal element is a composition where each part except the last is at least 2. The maximal element is composed of only 1s and 2s and ends in 1. Both sets are obviously of cardinality  $f_n$ , which is also the number of compositions of  $n$  into odd parts.

But, thanks to Eq. (3), we know that the elements

$$(10) \quad \pi^I(-1) = \pi_{i_1}(-1)\pi_{i_2}(-1) \cdots \pi_{i_r}(-1),$$

where  $i = (i_1, \dots, i_r)$  runs over compositions of  $n$  into odd parts, form a basis of  $\widetilde{\mathbf{Sym}}_n$ . Hence,

$$(11) \quad \widetilde{\mathbf{Sym}} = \mathcal{P}_n.$$

Also, since the commutative image of  $\pi_n(q)$  is  $\frac{1}{n}p_n$  for all  $q$ , this makes it clear that the commutative image of  $\widetilde{\mathbf{Sym}}$  is the subalgebra of  $Sym$  generated by odd power-sums  $p_{2k+1}$ .

To summarize, we have shown that the peak classes  $\Pi_P$  in  $\mathbf{Sym}$  form a linear basis of a graded Hopf subalgebra  $\mathcal{P}$  of  $\mathbf{Sym}$ , which is also a left ideal for the internal product, and we have described a basis of it, which is mapped onto products of odd power sums by the commutative image homomorphism. Since the  $\pi_n(-1)$  are Lie idempotents, this also determines the primitive Lie algebra of  $\mathcal{P}$  as the free Lie algebra generated by the  $\pi_{2k+1}(-1)$ . It is interesting to remark that all this has been obtained without much effort by setting  $q = -1$  in a few formulas of [12].

### 3. THE QUASI-SYMMETRIC SIDE

To recover Stembridge's algebra, we have to look at the dual of  $\mathcal{P}$ . Since  $\mathcal{P}$  can be regarded either as a homomorphic image of  $\mathbf{Sym}$  (under  $\theta_{-1}$ ) or as a subalgebra of  $\mathbf{Sym}$  (spanned by the peak classes  $\Pi_P$ ), the dual  $\mathcal{P}^*$  can be realized either as a subalgebra, or as a quotient of  $QSym$ .

Recall that we have a nondegenerate duality between  $QSym$  and  $\mathbf{Sym}$  defined by [9, 15]

$$(12) \quad \langle F_I, R_J \rangle = \delta_{IJ}.$$

Let us first consider the noncommutative peak algebra  $\mathcal{P} = \mathbf{Peak}^*$  as the image of the Hopf epimorphism  $\varphi = \theta_{-1}$ . Then, the adjoint map

$$(13) \quad \varphi^* : \mathcal{P}^* \longrightarrow QSym$$

is an embedding of Hopf algebras. The duality between  $\mathcal{P}$  and  $\mathcal{P}^*$  is given by

$$(14) \quad \langle \varphi(F), G \rangle = \langle F, \varphi^*(G) \rangle.$$

Hence, if we denote by  $\Pi_P^*$  the dual basis of  $\Pi_P$ , we have for any ribbon  $R_I$  with descent set  $D = \text{Des}(I)$

$$(15) \quad \langle \varphi^*(\Pi_P^*), R_I \rangle = \langle \Pi_P^*, \varphi(R_I) \rangle = \begin{cases} 2^{|P|+1} & \text{if } P \subseteq D\Delta(D+1) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, in its realization as a subalgebra of  $QSym$ ,  $\mathcal{P}^*$  is spanned by Stembridge's quasi-symmetric functions

$$(16) \quad \Theta_P = \varphi^*(\Pi_P^*) = 2^{|P|+1} \sum_{P \subseteq \text{Des}(I)\Delta(\text{Des}(I)+1)} F_I.$$

Note also that thanks to the identity  $(1+q)(1-q) = 1-q^2$ , the kernel of  $\varphi$  is seen to be the ideal of **Sym** generated by the  $S_n((1+q)A)_{q=-1}$  for  $n \geq 1$ . These are the  $\chi_n$  of [5].

Finally, we can also consider  $\mathcal{P}$  as an abstract algebra with basis  $(\Pi_P)$ , and define a monomorphism  $\psi : \mathcal{P} \rightarrow \mathbf{Sym}$  by

$$(17) \quad \psi(\Pi_P) = \sum_{HP(I)=P} R_I.$$

Then, its adjoint  $\psi^* : QSym \rightarrow \mathcal{P}^*$  is an epimorphism. The product map  $\vartheta = \varphi^* \circ \psi^* : \mathcal{P}^* \rightarrow \mathcal{P}^*$  has been considered by Stembridge [21], and its diagonalization is given in [5]. We can easily recover its properties from the results of the previous section, since clearly  $\vartheta = (\psi \circ \varphi)^*$  coincides with  $\theta_{-1}$ . Its eigenvalues are then the integers  $2^{\ell(\lambda)}$ , where  $\lambda$  runs over partitions into odd parts. The spectral projectors are again constructed from the idempotents  $\pi_\lambda(-1)$ . Precisely, the projector onto the eigenspace associated with the eigenvalue  $2^k$  of  $\vartheta$  in  $QSym_n$  is the adjoint of the endomorphism of **Sym** $_n$  given by  $F \mapsto F * U_k$  where  $U_k = \sum \pi_\lambda(-1)$ , the sum being over all odd partitions of  $n$  with exactly  $k$  parts. The dimensions of these eigenspaces can also be easily computed.

#### 4. MISCELLANEOUS RELATED RESULTS

Here are some more results related to the recent literature. We choose to include them here for completeness.

**4.1. Noncommutative tangent numbers.** By definition,  $\widetilde{\mathbf{Sym}}$  is generated by the  $\tilde{S}_n$ ,  $n \geq 1$ . If we set  $q = -1$  in [12, Prop. 5.2] we establish that  $\tilde{S}_n = 2H_n$  for  $n \geq 1$ , where  $H_0 = 1$  and

$$(18) \quad H_n = \sum_{k=0}^{n-1} R_{1^k, n-k}.$$

Then [9, Prop. 5.24] gives us that

$$(19) \quad H = \sum_{n \geq 0} H_n = (1 - \mathbf{t})^{-1}$$

where  $\mathbf{t}$  is the (left) noncommutative hyperbolic tangent

$$(20) \quad \mathbf{t} = \text{TH} = \sum_{k \geq 0} (-1)^k T_{2k+1}, \quad T_{2k+1} = R_{2k+1}.$$

Hence,  $\widetilde{\mathbf{Sym}}$  is contained in the subalgebra generated by the  $T_{2k+1}$ , and since we already know that the dimension of  $\widetilde{\mathbf{Sym}}_n$  is the number of odd compositions of  $n$ , we have in fact equality. Thus, the  $T^I = T_{i_1} \cdots T_{i_r}$  ( $I$  odd) form a multiplicative basis of  $\widetilde{\mathbf{Sym}}$  (this is the same as the basis  $\Gamma^P$  of [19]).

**4.2. Peak Lie idempotents.** We have seen in Section 2, Eq. (10), that the  $\pi^I(-1)$ ,  $I$  odd, form a basis of  $\widetilde{\mathbf{Sym}}$ , so that  $\widetilde{\mathbf{Sym}}_n$  contains Lie idempotents iff  $n$  is odd. In [19], the images  $\tilde{L}_n$  of some classical Lie idempotents  $L_n$  are calculated.

The images  $\tilde{\Psi}_n = \Psi_n((1-q)A)$  of the Dynkin elements  $\Psi_n$  are given in closed form for any  $q$  in [12, Prop. 5.34]. It suffices to set  $q = -1$  in this formula to obtain [19, Prop. 7.3].

On the other hand [19, Prop. 7.2] gives an interesting new formula for  $\tilde{\Phi}_n$ . Yet, the first part of the analysis can be simplified by applying Eq. (19) to the calculation of the generating series  $\log \tilde{\sigma}_1$ . Indeed,

$$\begin{aligned} \log \tilde{\sigma}_1 &= \log(1 + \mathbf{t}) - \log(1 - \mathbf{t}) \\ &= 2 \sum_{k \geq 0} \frac{\mathbf{t}^{2k+1}}{2k+1} \\ &= 2 \sum_{I \text{ and } \ell(I) \text{ odd}} \frac{(-1)^{(|I|-\ell(I))/2}}{\ell(I)} T^I. \end{aligned}$$

Finally, to obtain the image of Klyachko's idempotent  $\tilde{K}_n(q)$  (see [12, Prop. 6.3] for a definition of  $K_n(q)$ ) one has to set  $t = -1$  in [12, Prop. 8.2].

**4.3. Structure of the Peak algebras  $(\mathcal{P}_n, *)$ .** Using the construction of [12, Sec. 3.4] restricted to odd partitions  $\lambda$  of  $n$ , it follows from Eq. (3) that the idempotents  $E_\lambda(\pi(-1))$ , associated to the sequence  $\pi_n(-1)$ , form a complete set of orthogonal idempotents of  $\mathcal{P}_n$ . Regarding  $\mathcal{P}_n$  as a quotient of the descent algebra makes it clear that the left ideals  $\mathcal{P}_n * E_\lambda(\pi(-1))$  are the indecomposable projectives modules of  $\mathcal{P}_n$ . We obtain explicitly the multiplicative structure of  $(\mathcal{P}_n, *)$  by adapting [12, Lem. 3.10] to the sequence  $\pi_n(-1)$  (instead of  $\Psi_n$ ), and then imitating the rest of the argument presented there for the descent algebra.

**4.4. Hall-Littlewood basis.** The peak algebra  $\mathcal{P} = \text{Peak}^*$  can be regarded as a noncommutative version of the subalgebra of  $Sym$  spanned by the Hall-Littlewood functions  $Q_\lambda(X; -1)$ , where  $\lambda$  runs over strict partitions. Actually, it is easy to show that the noncommutative Hall-Littlewood functions of [4, 8] at  $q = -1$  yield two different analogous bases of  $\mathcal{P}$ . We do it here for [8] but a similar argument can be applied to [4].

Recall that the polynomials  $H_I(A; q)$  of [8] are defined as noncommutative analogues of the  $Q'_\mu = Q_\mu(X/(1-q); q)$ . To obtain the correct analogues of Schur's  $q$ -functions, one has to apply the  $(1-q)$  transform before setting  $q = -1$ .

**Proposition 4.1.** *The specialized noncommutative Hall-Littlewood functions*

$$(21) \quad \mathbb{Q}_I = H_J((1-q)A; q)_{q=-1},$$

where  $I$  runs over all peak compositions, form a basis of  $\mathcal{P}$ .

Indeed, the factorization of  $H$ -functions at roots of unity imply that

$$(22) \quad \mathbb{Q}_I = \mathbb{Q}_{i_1 i_2} \mathbb{Q}_{i_3 i_4} \cdots \mathbb{Q}_{i_{2k-1} i_{2k}} \mathbb{Q}_{i_{2k+1}}$$

(where  $i_{2k+1} = 0$  if  $I$  is of even length), and simple calculations yield

- $\mathbb{Q}_n = 2\Pi_\emptyset$ ,
- $\mathbb{Q}_{n-1,1} = 2(\Pi_{\{n-1\}} + \Pi_\emptyset)$ ,
- and for  $2 \leq k \leq n-2$ ,  $\mathbb{Q}_{k,n-k} = 4(\Pi_{\{k\}} + \Pi_{\{k+1\}} + \Pi_\emptyset)$ ,

where  $\Pi_P$  is defined in Eq. (5). From this, it is straightforward to prove that the family  $\mathbb{Q}_I$  is triangular with respect to the family  $\Pi_P$ , and hence the proposition follows.

## 5. REPRESENTATION THEORY OF THE 0-HECKE-CLIFFORD ALGEBRAS

The character theory of symmetric groups (in characteristic 0), as worked out by Frobenius, can be summarized as follows. Let  $R_n$  denote the free abelian group spanned by isomorphism classes of irreducible representations of  $\mathbb{C}\mathfrak{S}_n$ . Endow the direct sum

$$(23) \quad R = \bigoplus_{n \geq 0} R_n$$

with the addition corresponding to direct sum, and multiplication  $R_m \otimes R_n \rightarrow R_{m+n}$  corresponding to induction from  $\mathfrak{S}_m \times \mathfrak{S}_n$  to  $\mathfrak{S}_{m+n}$  via the natural embedding. The linear map sending the class of an irreducible representation  $[\lambda]$  to the Schur function  $s_\lambda$  is then a ring isomorphism between  $R$  and  $Sym$  (see, e.g., [14]). Moreover, we can define a structure of graded Hopf algebra on  $R$  with comultiplication corresponding to restrictions from  $\mathfrak{S}_n$  down to  $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$  and summing over  $k$ . The linear map above gives rise to an isomorphism of graded Hopf algebras.

It is known that the pair of graded Hopf algebras  $(\mathbf{Sym}, QSym)$  admits a similar interpretation, in terms of the tower of the 0-Hecke algebras  $H_n(0)$  of type  $A_{n-1}$  (see [13]). Recall that the (Iwahori-) Hecke algebra  $H_n(q)$  is the  $\mathbb{C}$ -algebra generated by elements  $T_i$  for  $i < n$  with the relations:

$$(24) \quad \begin{aligned} T_i^2 &= (q-1)T_i + q && \text{for } 1 \leq i \leq n-1, \\ T_i T_j &= T_j T_i && \text{for } |i-j| > 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leq i \leq n-2, \end{aligned}$$

(here, we assume that  $q \in \mathbb{C}$ ). The 0-Hecke algebra is obtained by setting  $q = 0$  in these relations. Then, the first relation becomes  $T_i^2 = -T_i$ . If we denote

by  $G_n = G_0(H_n(0))$  the Grothendieck group of the category of finite dimensional  $H_n(0)$ -modules, and by  $K_n = K_0(H_n(0))$  the Grothendieck group of the category of projective  $H_n(0)$ -modules, the direct sums  $\mathcal{G} = \bigoplus_{n \geq 0} G_n$  and  $\mathcal{K} = \bigoplus_{n \geq 0} K_n$ , endowed with the same operations as above, are respectively isomorphic with  $\overline{QSym}$  and **Sym**.

The aim of this final section is our main result: to provide a similar interpretation for the pair  $(\mathcal{P}, \mathcal{P}^*)$ . The relevant tower of algebras is the 0-Hecke-Clifford algebras, which are degenerate versions of Olshanski's Hecke-Clifford algebras.

**5.1. Hecke-Clifford algebra.** The complex Clifford algebra  $Cl_n$  is generated by  $n$  elements  $c_i$  for  $i \leq n$  with the relations

$$(25) \quad c_i c_j = -c_j c_i \quad \text{for } i \neq j \quad \text{and} \quad c_i^2 = -1.$$

For each subset  $D = \{i_1 < i_2 < \dots < i_k\} \subset \{1 \dots n\}$ , we denote by  $c_D$  the product

$$(26) \quad c_D := \prod_{i \in D}^{\rightarrow} c_i = c_{i_1} c_{i_2} \dots c_{i_k}.$$

It is easy to see that  $(c_D)_{D \subset \{1 \dots n\}}$  is a basis of the Clifford algebra.

The Hecke-Clifford superalgebra [17] is the unital  $\mathbb{C}$ -algebra generated by the  $c_i$ , and  $n - 1$  elements  $t_i$  satisfying the Hecke relations in the form

$$(27) \quad \begin{aligned} t_i^2 &= (q - q^{-1})t_i + 1 && \text{for } 1 \leq i \leq n - 1, \\ t_i t_j &= t_j t_i && \text{for } |i - j| > 1, \\ t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1} && \text{for } 1 \leq i \leq n - 2. \end{aligned}$$

and the cross-relations

$$(28) \quad \begin{aligned} t_i c_j &= c_j t_i && \text{for } i \neq j, j + 1, \\ t_i c_i &= c_{i+1} t_i && \text{for } 1 \leq i \leq n - 1, \\ (t_i + q^{-1})c_{i+1} &= c_i(t_i - q) && \text{for } 1 \leq i \leq n - 1. \end{aligned}$$

The Hecke-Clifford algebra has a natural  $\mathbb{Z}_2$ -grading, for which the  $t_i$  are even and the  $c_j$  are odd. Henceforth, it will be considered as a superalgebra.

Setting  $t_i = q^{-1}T_i$  and taking the limit  $q \rightarrow 0$  after clearing the denominators, we obtain the 0-Hecke-Clifford algebra  $HCl_n(0)$ , which is generated by the 0-Hecke algebra and the Clifford algebra, with the cross-relations

$$(29) \quad \begin{aligned} T_i c_j &= c_j T_i && \text{for } i \neq j, j + 1, \\ T_i c_i &= c_{i+1} T_i && \text{for } 1 \leq i \leq n - 1, \\ (T_i + 1)c_{i+1} &= c_i(T_i + 1) && \text{for } 1 \leq i \leq n - 1. \end{aligned}$$

Let  $\sigma = \sigma_{i_1} \dots \sigma_{i_p}$  be a reduced word for a permutation  $\sigma \in \mathfrak{S}_n$ . The defining relations of  $H_n(q)$  ensure that the element  $T_\sigma := T_{i_1} \dots T_{i_p}$  is independent of the chosen reduced word for  $\sigma$ . The family  $(T_\sigma)_{\sigma \in \mathfrak{S}_n}$  is a basis of the Hecke algebra. Thus a basis for  $HCl_n(q)$  is given by  $(c_D T_\sigma)_{D \subset \{1, \dots, n\}, \sigma \in \mathfrak{S}_n}$ , and consequently, the dimension of  $HCl_n(q)$  is  $2^n n!$  for all  $q$ .



**5.2. Quasi-symmetric Characters of induced modules.** Since  $H_n(0)$  is the subalgebra of  $HCl_n(0)$  generated by the  $T_i$ , our main tool in the sequel will be the induction process with respect to this inclusion. Let us recall some known facts about the representation theory of  $H_n(0)$ . There are  $2^{n-1}$  simple  $H_n(0)$ -modules. These are all one dimensional and can be conveniently labelled by compositions  $I$  of  $n$ . The structure of the simple module  $S_I := \mathbb{C} \epsilon_I$  is given by

$$(30) \quad T_j \epsilon_I = \begin{cases} -\epsilon_I & \text{if } j \in \text{Des}(I), \\ 0 & \text{otherwise.} \end{cases}$$

As described in [13], there is an isomorphism  $\text{ch}: \mathcal{G} \rightarrow QSym$  which we call the *Frobenius characteristic*. This maps the simple module  $S_I$  to the quasi-symmetric function  $F_I$ .

Let us define the  $HCl_n(0)$ -module  $M_I$  as the module induced by  $S_I$  through the natural inclusion map, that is

$$(31) \quad M_I := \text{Ind}_{H_n(0)}^{HCl_n(0)}(S_I) = HCl_n(0) \otimes_{H_n(0)} S_I.$$

A basis for  $M_I$  is given by  $(c_D \epsilon_I)_{D \subset \{1, \dots, n\}}$ . A basis element can be depicted conveniently as follows. The boxes of the ribbon diagram associated with  $I$  are numbered from left to right and from top to bottom. We put a “ $\times$ ” in the  $i$ -th box if  $i \in D$ . For example  $c_{\{1,3,4,6\}} \epsilon_{(2,1,3)} = c_1 c_3 c_4 c_6 \epsilon_{(2,1,3)}$  is depicted by

$$(32) \quad (2, 1, 3) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \quad c_{\{1,3,4,6\}} \epsilon_{(2,1,3)} = \begin{array}{|c|c|} \hline \times & \\ \hline & \times \\ \hline \times & & \times \\ \hline \end{array}$$

We can graphically view the set  $\text{Des}(I)$  as the set of boxes with a box below, and the set  $HP(I)$  as the set of boxes with boxes below and to the left. In the example above,  $\text{Des}(2, 1, 3) = \{2, 3\}$  are the boxes labeled 2 and 3, and  $HP(2, 1, 3) = \{2\}$  is only the box 2.

We now remark that  $T_i$  acts only on the  $i$ -th and  $i+1$ -st boxes. On the graphical representation, drawing only the boxes  $i$  and  $i+1$ , the rules (29) read

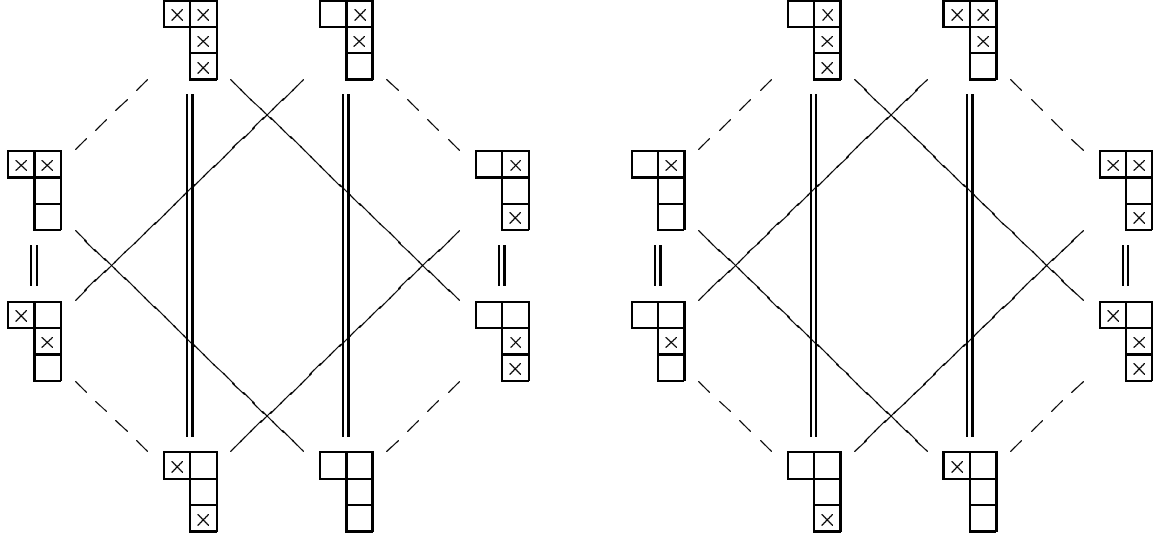
$$(33) \quad \begin{array}{l} T_i \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} = 0 \\ T_i \begin{array}{|c|c|} \hline \times & \square \\ \hline \end{array} = 0 \\ T_i \begin{array}{|c|} \hline \times \\ \hline \square \\ \hline \end{array} = - \begin{array}{|c|} \hline \square \\ \hline \times \\ \hline \end{array} \\ T_i \begin{array}{|c|} \hline \times \\ \hline \times \\ \hline \end{array} = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \end{array} \quad \begin{array}{l} T_i \begin{array}{|c|c|} \hline \square & \times \\ \hline \end{array} = - \begin{array}{|c|c|} \hline \square & \times \\ \hline \end{array} + \begin{array}{|c|c|} \hline \times & \square \\ \hline \end{array} \\ T_i \begin{array}{|c|c|} \hline \times & \times \\ \hline \end{array} = - \begin{array}{|c|c|} \hline \times & \times \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \\ T_i \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} = - \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \\ T_i \begin{array}{|c|} \hline \square \\ \hline \times \\ \hline \end{array} = - \begin{array}{|c|} \hline \square \\ \hline \times \\ \hline \end{array} \end{array}$$

At this point, we can make a couple of useful remarks. Looking at the support of the relation (33), we define

$$(34) \quad \begin{array}{|c|c|} \hline \square & \times \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline \times & \square \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \times & \times \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \times \\ \hline \square \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline \square \\ \hline \times \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \times \\ \hline \times \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}.$$

These relations can be interpreted as the cover relation of a (partial) order  $\leq_I$  on the subset  $D$  of  $\{1, \dots, n\}$ . Here is a picture of the Hasse diagram of this order for

the composition  $I = (2, 1, 1)$ . The poset clearly has two components corresponding to the two  $\mathbb{Z}_2$ -graded homogenous components of  $M_{(211)}$ .



The Hasse diagram of the order  $\leq_{(2,1,1)}$ .

The importance of this order comes from the following lemma, a direct consequence of Eq. (33).

**Lemma 5.1.** *The action of each  $T_i$  is triangular with respect to the order  $\leq_I$ , that is for all  $D$ ,*

$$(35) \quad T_i c_{D \in I} = \alpha(i, I, D) c_{D \in I} + \text{smaller terms}$$

with  $\alpha(i, I, D) \in \{0, -1\}$ .

The  $\alpha(i, I, D)$  are the eigenvalues of the  $T_i$ . They are equal to 0 in the leftmost columns of Eq. (33) and to  $-1$  in the rightmost ones.

A second consequence of Eq. (33) is that in a vertical (two boxes) diagram, the eigenvalue depends only on the content of the upper box whereas in a horizontal diagram it depends only on the content of the rightmost box. Thus the content of the boxes without a box below or on the left does not matter for computing the eigenvalues. This can be translated into the following lemma.

**Lemma 5.2.** *Suppose that  $k \in \{1 \dots n\}$  is such that  $k$  is not a descent of  $I$  and has no box to its left. Then for all  $D$ , the eigenvalues  $\alpha(i, I, D)$  satisfy*

$$(36) \quad \alpha(i, I, D) = \alpha(i, I, \{k\} \cup D).$$

Such a  $k$  is called a valley of the composition  $I$ .

Note that 1 and  $n$  can be valleys. There is obviously one more valley than the number of peaks.

Thanks to the order  $\leq_I$ , one can easily describe the structure of the restriction of  $M_I$  to  $H_n(0)$ . Our first goal is to get a composition series of  $\text{Res}_{H_n(0)} M_I$  in order to

compute its Frobenius characteristic. This can be done as follows. Let us choose a linear extension  $D_1, D_2 \dots D_{2^n}$  of  $\leq_I$ . For  $k \geq 1$ , define

$$(37) \quad M_I^k = \bigoplus_{l \leq k} \mathbb{C} c_{D_k} \epsilon_I,$$

and  $M_I^0 := \{0\}$ . Then, thanks to Lemma 5.1,  $M_I^k$  is clearly a sub-module of  $\text{Res}_{H_n(0)} M_I$ , and

$$(38) \quad \{0\} = M_I^0 \subset M_I^1 \subset M_I^2 \subset \dots \subset M_I^{2^n} = \text{Res}_{H_n(0)} M_I$$

is a composition series of the module  $\text{Res}_{H_n(0)} M_I$ . Let us compute the simple composition factors of the module  $S_{D_i, I} = \mathbb{C} \epsilon_{K(D_i, I)} := M_I^i / M_I^{i-1}$ . For  $1 \leq k < n$ , the generator  $T_k$  acts as  $T_k \epsilon_{K(D_i, I)} = \alpha(k, I, D_i) \epsilon_{K(D_i, I)}$ . The eigenvalue  $\alpha(k, I, D_i)$  equals  $-1$  if

$$(39) \quad (k+1 \in D_i \text{ and } k \notin \text{Des}(I)) \quad \text{or} \quad (k \notin D_i \text{ and } k \in \text{Des}(I)),$$

and 0 otherwise. Hence, according to Eq. (30),  $\text{ch}(S_{D_i, I}) = F_K$  where  $\text{Des}(K) = \text{Des}(K(D_i, I)) = \{1 \leq k < n \mid \alpha(k, I, D_i) = -1\}$ . When  $p$  is a peak of  $I$ , that is  $p \neq 1$ ,  $p-1 \notin \text{Des}(I)$  and  $p \in \text{Des}(I)$ , then  $|\{p-1, p\} \cap \text{Des}(K)| = 1$ . Indeed, if  $p \in D_i$  then  $p-1 \in \text{Des}(K)$  and  $p \notin \text{Des}(K)$  and if  $p \notin D_i$  then  $p-1 \notin \text{Des}(K)$  and  $p \in \text{Des}(K)$ . Thus,  $P = HP(I) \subseteq \text{Des}(K) \Delta(\text{Des}(K) + 1)$ . Moreover, for  $k \notin P \cup (P-1)$ , we can always find a  $D_i$  such that  $k \in \text{Des}(K(D_i, I))$ . All  $K$  such that  $P \subseteq \text{Des}(K) \Delta(\text{Des}(K) + 1)$  can be obtained, and thanks to Lemma 5.2 there are  $2^{|P|+1}$  sets  $D_i$  giving the same  $F_K$ . Thus, we have proved the following proposition.

**Proposition 5.3.** *The Frobenius characteristic of  $\text{Res}_{H_n(0)} M_I$  depends only on the peak set  $P$  of the composition  $I$  and is given by Stembridge's  $\Theta$  function*

$$(40) \quad \text{ch}(\text{Res}_{H_n(0)} M_I) = \Theta_P = 2^{|P|+1} \sum_{P \subseteq \text{Des}(K) \Delta(\text{Des}(K)+1)} F_K.$$

■

**5.3. Homomorphisms between induced modules.** The previous proposition suggests that  $\text{Res}_{H_n(0)} M_I$  is isomorphic to  $\text{Res}_{H_n(0)} M_J$  iff  $I$  and  $J$  have the same peak sets. This is actually true, and in fact,  $M_I$  and  $M_J$  are even isomorphic as  $HCl_n(0)$ -supermodules, as we will establish now.

**Theorem 5.4.** *Let  $I$  be a composition with valley set  $V$ , and let  $Cl_V$  be the subalgebra of  $Cl_n$  generated by  $(c_v)_{v \in V}$ . For  $c \in Cl_V$  define a map  $f_c$  from  $M_I$  to itself by*

$$(41) \quad f_c(x \epsilon_I) = x c \epsilon_I \quad \text{for all } x \in Cl_n.$$

*Then  $c \mapsto f_c$  defines a right action of  $Cl_V$  on  $M_I$  which commutes with the left  $HCl_n(0)$ -action. Moreover the map  $c \mapsto f_c$  is a graded isomorphism from  $Cl_V$  to  $\text{End}_{HCl_n(0)}(M_I)$ .*

*Proof* – Since  $M_I$  is freely generated as a  $Cl_n$ -module by  $\epsilon_I$ , a morphism  $f \in \text{End}_{HCl_n(0)}(M_I)$  is determined by  $f(\epsilon_I) = x\epsilon_I$  for  $x \in Cl_n$ . On the other hand, for  $x \in Cl_n$ , a map  $f_x(\epsilon_I) = x\epsilon_I$  is in  $\text{End}_{HCl_n(0)}(M_I)$  if and only if  $T_j f_x(\epsilon_I) = T_j x\epsilon_I = xT_j\epsilon_I$  for all  $1 \leq j < n$ . Thus, to prove the theorem, it is sufficient to see that  $f_x \in \text{End}_{HCl_n(0)}(M_I)$  if and only if  $x \in Cl_V$ . Equivalently,

$$(42) \quad \text{for all } 1 \leq j < n \quad T_j x\epsilon_I = \begin{cases} -x\epsilon_I & \text{if } j \in \text{Des}(I), \\ 0 & \text{otherwise,} \end{cases}$$

if and only if  $x \in Cl_V$ .

Let us first assume that  $x = c_D$  for  $D \subseteq V$ . This means that in the graphical representation of  $x\epsilon_I$  there is no box below nor to the left of a box with a “ $\times$ ”. If  $j \in \text{Des}(I)$ , the lower two equations of the right column of Eq. (33) then show that  $T_j x\epsilon_I = -x\epsilon_I$ . The top two equations on the left show that if  $j \notin \text{Des}(I)$  then  $T_j x\epsilon_I = 0$ . By linearity, we get that if  $x \in Cl_V$  then Eq. (42) holds. Conversely, let  $x \in Cl_n$  satisfy Eq. (42). Let  $c_D\epsilon_I$  be in the support of  $x$ , minimal with respect to  $\leq_I$ . If  $D \not\subseteq V$  then there is a box  $j$  with a “ $\times$ ” and a box below or to the left. Using Lemma 5.1 and Eq. (33) this would be a contradiction to Eq. (42). Hence  $c_D \in Cl_V$ . We can subtract it from  $x$  and repeat the argument above recursively to conclude that  $x \in Cl_V$ .  $\blacksquare$

**Theorem 5.5.** *The induced supermodules  $M_I$  and  $M_J$  are isomorphic if and only if the peak sets of  $I$  and  $J$  coincide.*

*Proof.* One direction of this theorem is implied by the previous section. If  $M_I$  is isomorphic to  $M_J$ , then we must have that  $\text{Res}_{H_n(0)} M_I$  is isomorphic to  $\text{Res}_{H_n(0)} M_J$ . In particular, they must have the same Frobenius characteristic. Thanks to Prop. 5.3,  $\text{ch}(\text{Res}_{H_n(0)} M_I)$  depends only on the peak set of  $I$ . Thus, if  $M_I$  and  $M_J$  are isomorphic then  $I$  and  $J$  have the same peak sets

The converse will follow once we construct explicit isomorphisms between any modules  $M_I$  and  $M_J$  in the same peaks class ( $HP(I) = HP(J)$ ), such that  $I$  and  $J$  differ exactly by one descent. Isomorphisms between any modules  $M_I$  and  $M_J$  in the same peaks class in general will be obtained by composition of the constructed ones.

Let  $I = J \cap \{k\}$  be such that  $HP(I) = HP(J)$ . Graphically, there are two possible cases to consider:

$$(43) \quad J = \begin{array}{c} \dots \\ \square \\ \square \square \\ \dots \end{array} \quad I = \begin{array}{c} \dots \\ \square \\ \square \\ \square \\ \dots \end{array}$$

or

$$(44) \quad J = \begin{array}{c} \dots \\ \square \square \square \\ \dots \end{array} \quad I = \begin{array}{c} \dots \\ \square \\ \square \square \\ \dots \end{array}$$

In the case (43), we construct a map  $f$  which sends  $\epsilon_I \mapsto \eta = (c_{\{k,k+1\}} - 1)\epsilon_J$ . We remark that both  $\eta$  and  $\epsilon_I$  are even. Furthermore,

$$(45) \quad \text{for all } 1 \leq i < n \quad T_i \eta = \begin{cases} -\eta & \text{if } i \in \text{Des}(I), \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, for  $i \notin \{k-1, k\}$ , the  $T_i$  commute with  $c_{\{k,k+1\}}$ . Moreover  $i \in \text{Des}(I)$  if and only if  $i \in \text{Des}(J)$ , hence the Eq. (45) follows in these cases. If  $i = k-1 \in \text{Des}(I)$ , then  $T_{k-1}\eta = (-c_{\{k-1,k+1\}}T_k + c_{\{k-1,k+1\}} - c_{\{k,k+1\}} - T_k)\epsilon_J = -\eta$ , and if  $i = k \in \text{Des}(I)$ , then  $T_k\eta = (-c_{\{k,k+1\}}T_k - c_{\{k,k+1\}} + 1 - T_k)\epsilon_J = -\eta$ . This allows us to define a non-trivial  $HCl_n(0)$  supermorphism  $f: M_I \rightarrow M_J$  where  $f(c_D\epsilon_I) = c_D\eta$ . Thanks to Eq. (45), the submodule spanned by  $\eta$  in  $M_J$  is isomorphic to  $M_I$ . But since both spaces have the same dimension we have that  $f$  is an isomorphism.

For the case (44), we proceed in the same way, sending  $\epsilon_J \mapsto (c_{\{k,k+1\}} + 1)\epsilon_I$ . This constructs a graded isomorphism from  $M_J$  to  $M_I$ .  $\blacksquare$

**5.4. Simple supermodules of  $HCl_n(0)$ .** We are now in a position to construct the simple supermodules of  $HCl_n(0)$ . Our approach is similar to Jones and Nazarov [10]. Let  $I$  be a composition with peak set  $P$  and valley set  $V = \{v_1, v_2, \dots, v_k\}$ . Choose a minimal even idempotent of the Clifford superalgebra  $Cl_V$ . For example

$$(46) \quad e_I := \frac{1}{2^l} (1 + \sqrt{-1}c_{v_1}c_{v_2})(1 + \sqrt{-1}c_{v_3}c_{v_4}) \dots (1 + \sqrt{-1}c_{v_{2l}}c_{v_{2l+1}}),$$

where  $l := \lfloor \frac{k}{2} \rfloor = \lfloor \frac{|P|+1}{2} \rfloor$ . Define  $HClS_I := Cl_n e_I \epsilon_I$  as the  $HCl_n(0)$ -module generated by  $e_I \epsilon_I$ . One has to show that  $HClS_I$  does not depend on the chosen minimal idempotent  $e_I$ , but this is an easy consequence of the representation theory of  $Cl_V$  which is known to be supersimple (see e.g. [11]). Suppose that  $e_I$  and  $e'_I$  are two minimal even idempotents of  $Cl_V$ . Since  $Cl_V$  is supersimple, there exist  $x, y, x', y' \in Cl_V$  such that  $e_I = x'e'_I y'$  and  $e'_I = x e_I y$ . Then by the representation theory of  $Cl_V$ , we know that  $f_y$  and  $f_{y'}$  are two mutually reciprocal isomorphisms between  $Cl_V e_I$  and  $Cl_V e'_I$  and hence between  $Cl_n e_I \epsilon_I$  and  $Cl_n e'_I \epsilon_I$ .

When  $n$  is even  $Cl_V e_I$  has dimension  $2^{\frac{|P|+1}{2}}$  and when  $n$  is odd the dimension is  $2^{\frac{|P|+1}{2}-1}$ . In short we can write  $2^{\lfloor \frac{|P|+1}{2} \rfloor}$  for the dimension in both cases. Thus  $HClS_I$  has dimension  $2^{n - \lfloor \frac{|P|+1}{2} \rfloor}$ .

A direct corollary to Theorem 5.4 is the following.

**Corollary 5.6.** *The induced module  $M_I$  is the direct sum of  $2^{\lfloor \frac{|P|+1}{2} \rfloor}$  isomorphic copies of  $HClS_K$ , where  $K$  is the peak composition associated to  $I$ .  $\blacksquare$*

We are now ready to show our main theorem and define the Frobenius characteristic between  $HCl_n(0)$  modules and  $\mathcal{P}^*$ , which we again denote by  $\text{ch}$ .

**Theorem 5.7.** *The set  $\{HClS_I := Cl_n e_i \epsilon_I\}$ , where  $I$  runs over all peak compositions, is a complete set of pairwise non-isomorphic simple supermodules of  $HCl_n(0)$ . Moreover, there is a graded Hopf isomorphism defined by*

$$(47) \quad \begin{aligned} \text{ch}: \quad \tilde{\mathcal{G}} &\longrightarrow \mathcal{P}^* \\ HClS_I &\longrightarrow 2^{-\lfloor \frac{|P|+1}{2} \rfloor} \Theta_{HP(I)} \end{aligned}$$

where  $HP(I)$  is the peak set of  $I$ , and  $\tilde{\mathcal{G}} = \bigoplus_{n \geq 0} G_0(HCl_n(0))$ .

Thus the  $(1 - q)$ -transform at  $q = -1$  can be interpreted as the induction map from  $G_0(H_n(0))$  to  $G_0(HCl_n(0))$ . This maps  $\mathcal{G}$  to  $\tilde{\mathcal{G}}$ .

*Proof* – Suppose that  $S$  is a simple supermodule of  $HCl_n(0)$ . Decompose the  $H_n(0)$ -socle of  $S$  into simple modules and choose a non-zero vector  $v$  in one of these simple factors. Then  $v$  is a common eigenvector of all the  $T_i$ , so that there is a  $I$  such that  $\epsilon_I \mapsto v$  defines a  $H_n(0)$ -morphism  $\phi : S_I \rightarrow S$ . Then, since  $S$  is supersimple,  $v$  generates  $S$  under the action of  $HCl_n(0)$ . Thus by induction, there is a surjective morphism  $M_I \rightarrow S$ . Hence, each simple module of  $HCl_n(0)$  must be a quotient of some  $M_I$  and consequently of some  $HClS_I$ .

Now, we know that given two  $HClS_I$ , either they are isomorphic (when they have the same peak sets) or else there is no morphism between them. Thus  $HClS_I$  has to be simple. The multiplication and comultiplication structures are induced from  $\mathcal{G}$  and  $QSym$  and the Frobenius characteristic between them.  $\blacksquare$

By duality, we obtain  $\text{ch}^*: \mathcal{P} \rightarrow \tilde{\mathcal{G}}^*$  where  $\tilde{\mathcal{G}}^* = \bigoplus_{n \geq 0} K_0(HCl_n(0))$ . We also remark that the dimension of the superradical of  $HCl(0)$  is thus

$$(48) \quad 2^n n! - \sum_P 2^{2n - (|P|+1)}.$$

where the sum is over all peak sets of  $n$ . It is still open to find nice generators for this radical.

**5.5. Decomposition matrices.** The generic Hecke-Clifford algebra  $HCl_n(q)$  has its simple modules  $U_\lambda$  labelled by partitions into distinct parts, and under restriction to  $H_n(q)$ ,  $U_\lambda$  has as Frobenius characteristic

$$(49) \quad \text{ch}(U_\lambda) = 2^{-\lfloor \ell(\lambda)/2 \rfloor} Q_\lambda$$

where  $Q_\lambda$  is Schur's  $Q$ -function (see [17, 10]).

By Stembridge's formula [21], we have

$$(50) \quad \text{ch}(U_\lambda) = 2^{-\lfloor \ell(\lambda)/2 \rfloor} \sum_{T \in \mathcal{ST}^\lambda} \Theta_{\Lambda(T)}$$

where  $\mathcal{ST}^\lambda$  is the set of standard shifted tableaux of shape  $\lambda$ , and  $\Lambda(T)$  the peak set of  $T$ . The quasisymmetric characteristics of the simple  $HCl_n(0)$  modules are

proportional to the  $\Theta$ -functions, and the coefficients  $d_{\lambda I}$  in the expression

$$(51) \quad \text{ch}(U_\lambda) = \sum_I d_{\lambda I} \text{ch}(HClS_I)$$

form the decomposition matrices of the Hecke-Clifford algebras at  $q = 0$ .

For  $\lambda$  a strict partition of  $n$ , and  $I$  a peak composition of  $n$  with peak set  $P$ , one has explicitly

$$(52) \quad d_{\lambda I} = 2^{\lfloor \frac{1}{2}\ell(I) \rfloor - \lfloor \frac{1}{2}\ell(\lambda) \rfloor} |\{T \in ST^\lambda \mid \Lambda(T) = P\}|.$$

This is the analog for  $HCl_n(0)$  of Carter's combinatorial formula for the decomposition numbers of  $H_n(0)$  [7].

Here are the decomposition matrices for  $n \leq 9$ . Note that for  $n = 2, 3$ ,  $HCl_n(0)$  is semi-simple.

$$\begin{array}{c} 3 \\ 21 \end{array} \begin{bmatrix} 3 & 21 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} 4 \\ 31 \end{array} \begin{bmatrix} 4 & 31 & 22 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c} 5 \\ 41 \\ 32 \end{array} \begin{bmatrix} 3 & 41 & 32 & 23 & 221 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} 6 \\ 51 \\ 42 \\ 321 \end{array} \begin{bmatrix} 6 & 51 & 42 & 33 & 321 & 24 & 231 & 222 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} 7 \\ 61 \\ 52 \\ 43 \\ 421 \end{array} \begin{bmatrix} 7 & 61 & 52 & 43 & 421 & 34 & 331 & 322 & 25 & 241 & 232 & 223 & 2221 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{c} 8 \\ 71 \\ 62 \\ 53 \\ 521 \\ 431 \end{array} \begin{bmatrix} 8 & 71 & 62 & 53 & 521 & 44 & 431 & 422 & 35 & 341 & 332 & 323 & 3221 & 26 & 251 & 242 & 233 & 2321 & 224 & 2231 & 2222 \\ 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 1 & 1 & 2 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 1 & 4 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 4 & 4 \end{bmatrix}$$

	9	81	72	63	621	54	531	522	45	441	432	423	4221	36	351	342	333	3321	324	3231	3222	27	261	252	243	2421	234	2331	2322	225	2241	2232	2223	22221		
9	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
81	0	1	1	1	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
72	0	0	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0	1	0	0	0	1	1	1	0	1	0	0	1	0	0	1	0	0	0	0
63	0	0	0	1	0	1	1	1	1	1	2	1	2	0	1	2	2	2	1	2	2	0	0	1	1	2	1	2	2	0	2	2	2	2	0	
621	0	0	0	0	1	0	1	2	0	1	2	2	2	0	1	2	2	2	2	2	2	0	0	1	1	2	1	2	2	1	2	2	2	2	0	
54	0	0	0	0	0	1	0	0	0	1	1	1	0	0	0	1	1	2	0	2	2	0	0	0	1	0	0	2	2	0	0	2	0	2	0	
531	0	0	0	0	0	0	1	1	0	1	3	2	4	0	0	2	3	6	1	6	8	0	0	0	1	2	1	4	6	0	2	6	4	4	4	
432	0	0	0	0	0	0	0	0	0	1	0	2	0	0	0	1	2	0	2	4	0	0	0	0	0	0	0	2	2	0	0	2	2	2	2	

REFERENCES

- [1] M. AGUIAR, N. BERGERON, and K. NYMAN, *The peak algebra and the descent algebra of type B and D*, to appear in Trans. of the AMS. (math.CO/0302278.)
- [2] M. AGUIAR, N. BERGERON, and F. SOTTILE, *Combinatorial Hopf Algebra and generalized Dehn-Sommerville relations*, to appear. (math.CO/0310016.)
- [3] N. BERGERON, S. MYKYTIUK, F. SOTTILE and S. VAN WILLIGENBURG, *Non-commutative Pieri operations on posets* J. Comb. Theory, Ser. A, **91** (2000), 84–110.
- [4] N. BERGERON and M. ZABROCKI, *q and q, t-Analogs of Non-commutative Symmetric Functions*, To appear in Disc. Math. (math.CO/0106255).
- [5] L.J. BILLERA, S.K. HSIAO and S. VAN WILLIGENBURG, *Quasisymmetric functions and Eulerian enumeration*, preprint, 2002.
- [6] D. BLESSENOHL and H. LAUE, *Algebraic combinatorics related to the free Lie algebra*, Actes du 29ème Séminaire Lotharingien, Publ. IRMA, Strasbourg (1993), 1–21.
- [7] R.W. CARTER, *Representation theory of the 0-Hecke algebra*, J. Algebra **15** (1986), 89–103.
- [8] F. HIVERT, *Hecke algebras, difference operators, and quasi-symmetric functions*, Adv. in Math. **155** (2000), 181–238.
- [9] I.M. GELFAND, D. KROB, A. LASCoux, B. LECLERC, V.S. RETAKH and J.-Y. THIBON, *Noncommutative symmetric functions*, Adv. in Math. **112** (1995), 218–348.
- [10] A.R. JONES and M.L. NAZAROV, *Affine Sergeev algebra and q-analogues of the Young symmetrizers for projective representations of symmetric groups*, Proc. London Math. Soc (3) **78** (1999), 481–512.
- [11] T. JÓZEFIAK, *Characters of projective representations of symmetric groups*, Expositiones Math. **7** (1989), 193–247.
- [12] D. KROB, B. LECLERC and J.-Y. THIBON, *Noncommutative symmetric functions II: Transformations of alphabets*, Internal J. Alg. Comput. **7** (1997), 181–264.
- [13] D. KROB and J.-Y. THIBON, *Noncommutative symmetric functions IV: Quantum linear groups and hecke algebras at q = 0*, J. Alg. Comb. **6** (1997), 339i–376.
- [14] I.G. MACDONALD, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995.
- [15] C. MALVENUTO and C. REUTENAUER, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), 967–982.
- [16] K. NYMAN, *The peak algebra of the symmetric groups*, preprint.
- [17] G. OLSHANSKI, *Quantized universal enveloping superalgebra of type Q and a super-extension of the Hecke algebra*, Lett. Math. Phys. **24** (1992), 93–102.
- [18] C. REUTENAUER, *Free Lie algebras*, Oxford University Press, 1993.
- [19] M. SCHOCKER, *The peak algebra of the symmetric group revisited*, preprint math.RA/0209376.
- [20] I. SCHUR, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. reine angew. Math. **139** (1911), 155–250.
- [21] J. STEMBRIDGE, *Enriched P-partitions*, Trans. Amer. Math. Soc. **349** (1997), 763–788.



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