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Multipartite P-Partitions and Inner Products of Skew Schur Functions

Ira M. Gessel[1]

ABSTRACT. A generalization of Richard Stanley's theory of P-partitions to multipartite partitions is developed, which is used to count r-tuples of permutations whose product is a given permutation according to their descent sets. The theory gives a combinatorial interpretation to numbers associated with inner products of certain skew Schur functions.

1. INTRODUCTION. Let π be a permutation of $[n] = \{1,2,\cdots, n\}$. The descent set of π is defined to be $D(\pi) = \{i|\pi(i) > \pi(i+1)\}$. A general problem of permutation enumeration is to find the number of permutations in a given set of permutations with a given descent set. We shall be concerned here primarily with the set of permutations which extend some poset to a total order. In section 2 we review some of Richard Stanley's theory of P-partitions [13] and show how a special case leads to formulas involving symmetric functions. In particular we derive a formula equivalent to that of Foulkes [3] for the number of permutations π for which $D(\pi)$ and $D(\pi^{-1})$ have specified values.

In section 3 we consider the following problem: given a permutation π in the symmetric group S_n and subsets $A_1,\cdots, A_r$ of $[n-1]$, how many r-tuples $\sigma_1,\cdots, \sigma_r$ of permutations are there with $\sigma_1\sigma_2\cdots\sigma_r = \pi$ and $D(\sigma_i) = A_i$? The solution of this problem leads to a combinatorial interpretation for the scalar product of an arbitrary skew Schur function with an inner product of skew Schur functions of skew hook shape.

A more comprehensive account of the applications of P-partitions and multipartite partitions to permutation enumeration will be given elsewhere.

2. P-PARTITIONS. Let P be an arbitrary partial order of $[n]$. We write $<_P$ for the partial order of P, and $<$ for the usual total order on $[n]$. Let X be an infinite totally ordered set (for example, the positive integers). A P-partition is a function $f:[n] \to X$ satisfying

1) $i <_P j$ implies $f(i) \le f(j)$

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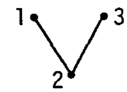
2) $i <_P j$ and $i > j$ implies $f(i) < f(j)$.

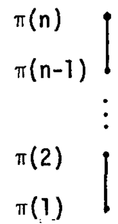
We denote the set of P-partitions by $A(P)$. Thus, if P is the poset



then $A(P)$ is the set of functions $f:[3] \to X$ satisfying $f(2) < f(1)$ and $f(2) \le f(3)$.

The definition of P-partitions is due to Stanley [13]; see also Knuth [7]. (Stanley defined P-partitions to be order-reversing rather than order-preserving.)

If π is a permutation of $[n]$ we may identify π with the total order



in which $\pi(i) <_\pi \pi(j)$ iff $i < j$. Then $A(\pi)$ is the set of functions $f: [n] \to X$ satisfying $f(\pi(1)) \sim_1 f(\pi(2)) \sim_2 \cdots \sim_{n-1} f(\pi(n))$, where $\sim_i$ is $\le$ if $\pi(i) < \pi(i+1)$ and $\sim_i$ is $<$ if $\pi(i) > \pi(i+1)$.

We now define $L(P)$ to be the set of permutations of $[n]$ which extend P to a total order; that is, π is in $L(P)$ iff $i <_P j$ implies $\pi^{-1}(i) < \pi^{-1}(j)$. The connection between $L(P)$ and $A(P)$ is given by the following theorem, which was proved in some special cases by MacMahon [11, Vol. 2, pp. 188-212], for "naturally labeled posets" by Knuth [7], and in general by Stanley [13]. All unions in this paper are disjoint.

THEOREM 1. $A(P) = \bigcup_{\pi \in L(P)} A(\pi)$.

PROOF. We proceed by induction on the number of incomparable pairs of elements in P. If there are none, then P is a total order and the theorem is trivial. Now suppose that i and j are incomparable in P, with $i < j$. Let P_{ij} be the poset obtained from P by adding the relation $i <_P j$ and let P_{ji} be defined similarly. It is easily seen that $L(P) = L(P_{ij}) \cup L(P_{ji})$ and $A(P) = A(P_{ij}) \cup A(P_{ji})$. Thus the theorem follows by induction.

To every x in X we assign a weight $w(x)$, which will always be a monomial in a power series ring. It is convenient to identify x with its weight. In this section we assume that the weights of elements of X are algebraically independent. We define the weight of a function $f: [n] \to X$ to be $w(f) = w(f(1))\, w(f(2)) \cdots w(f(n))$ and we define the generating function for P to be

$$\Gamma(P) = \sum_{f \in A(P)} w(f).$$

It follows from Theorem 1 that

$$\Gamma(P) = \sum_{\pi \in \mathcal{L}(P)} \Gamma(\pi). \tag{1}$$

Thus the power series of the form $\Gamma(\pi)$ will be important. To study them we introduce some notation. To any subset $A = \{a_1 < a_2 < \cdots < a_k\}$ of $[n-1]$ we may associate the composition $(a_1, a_2 - a_1, \cdots, a_k - a_{k-1}, n - a_k)$ of n. Then for any permutation π in S_n we define the descent composition $C(\pi)$ to be the composition associated to the descent set $D(\pi)$. It is convenient to transfer the partial order of inclusion of subsets to compositions (where it is reverse refinement).

It is clear that $\Gamma(\pi)$ depends only on $C(\pi)$. Thus we may define formal power series F_L, indexed by compositions, by $F_{C(\pi)} = \Gamma(\pi)$. For example,

$$F_{12} = \sum_{x_1 < x_2 \le x_3} x_1 x_2 x_3 .$$

F_L is not symmetric in the elements of X unless $L = 1^n$ or $L = n$. However F_L does have the property that if $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_m$ for $x_i, y_j \in X$ then the coefficient of $x_1^{k_1} \cdots x_m^{k_m}$ in F_L is equal to the coefficient of $y_1^{k_1} \cdots y_m^{k_m}$. We call power series in $\mathbb{Z}[[X]]$ with this property quasisymmetric. We denote by Qsym the ring of quasisymmetric power series, and by Qsym_n the $\mathbb{Z}$-module of quasisymmetric power series homogeneous of degree n.

If $L = (L_1, \cdots, L_k)$ is a composition of n, we define M_L by

$$M_L = \sum_{x_1 < \cdots < x_k} x_1^{L_1} \cdots x_k^{L_k} .$$

It is clear that the M_L are a basis for Qsym_n. Moreover, by the definition of F_L and the partial order on compositions, we have

$$F_L = \sum_{K \ge L} M_K , \tag{2}$$

so by inclusion-exclusion,

$$M_L = \sum_{K \ge L} (-1)^{|K| - |L|} F_K , \tag{3}$$

where $|L|$ is the number of parts of L. Thus the F_L form a basis for

$Qsym_n$. It follows that $\Gamma(P)$ can be expressed uniquely as a linear combination of the F_L and hence the number of π in $L(P)$ with $C(\pi) = L$ is determined by $\Gamma(P)$. This number can be computed by inclusion-exclusion from the following:

COROLLARY 2. If $L = (L_1, \cdots, L_k)$ is a composition of n and $x_1 < \cdots < x_n$ then the coefficient of $x_1^{L_1} \cdots x_k^{L_k}$ in $\Gamma(P)$ is the number of permutations π in $L(P)$ with $C(\pi) \le L$.

PROOF. By Theorem 1, we need only consider the case where P is a permutation. In this case the assertion follows from (2).

If $\Gamma(P)$ is symmetric, we can use the machinery of symmetric functions to express the number of permutations π in $L(P)$ with $C(\pi) = L$ in a more concise form. We shall need some basic facts about symmetric functions. For more details, see Macdonald [10].

If $\lambda = (\lambda_1 \ge \cdots \ge \lambda_k > 0)$ is a partition, we define the monomial symmetric function m_λ by

$$m_\lambda = \sum x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k} ,$$

where the sum is over distinct monomials with the x_i distinct.
Note that

$$m_\lambda = \sum_L M_L \tag{4}$$

where the sum is over all rearrangements L of the parts of λ. (Thus $m_{221} = M_{122} + M_{212} + M_{221}$.) The complete homogeneous symmetric functions are defined by

$$h_n = \sum_{x_1 \le \cdots \le x_n} x_1 x_2 \cdots x_n$$

and $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}$. More generally, we set $h_L = h_{L_1} h_{L_2} \cdots h_{L_k}$ for any composition $L = (L_1, \cdots, L_k)$. Let Sym_n be the $\mathbb{Z}$-module of symmetric functions which are homogeneous of degree n (with integer coefficients). Then the m_λ and the h_λ, where λ ranges over partition of n, are bases for Sym_n. There is a symmetric scalar product on symmetric functions for which

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} . \tag{5}$$

Thus if g is a symmetric function, the coefficient of $x_1^{L_1} \cdots x_k^{L_k}$ in g is $\langle g, h_L \rangle$.

Now let us define symmetric functions S_L, where L is a composition, by

$$S_L = \sum_{K \le L} (-1)^{|L|-|K|} h_K .$$

Thus $S_{(2,1,3)} = h_{(2,1,3)} - h_{(3,3)} - h_{(2,4)} + h_{(6)}$. As we shall see later, these symmetric functions are skew Schur functions corresponding to skew hook shapes. They were first considered by MacMahon [11, Vol. 1, pp. 199-202], who wrote h_L for our S_L. He showed that S_L is the sum of all $x_1 x_2 \cdots x_n$ where, if A is the subset of $[n-1]$ corresponding to L,

$$x_i \le x_{i+1} \quad \text{if} \quad i \notin A$$

$$x_i > x_{i+1} \quad \text{if} \quad i \in A .$$

Thus $S_L = \Gamma(P_L)$ where P_L is the poset defined by

$$i <_P i+1 \quad \text{if} \quad i \notin A$$

$$i+1 <_P i \quad \text{if} \quad i \in A .$$

If g is a symmetric function then g is quasisymmetric and hence is a linear combination of the F_L.

THEOREM 3. Let g be a symmetric power series in the elements of X. Then $g = \sum_L a_L F_L$, where $a_L = \langle g, S_L \rangle$.

PROOF. Suppose that $g = \sum_\lambda b_\lambda m_\lambda$. Then by (5), $b_\lambda = \langle g, h_\lambda \rangle$. Thus by (4), $g = \sum_K b_K M_K$, where $b_K = \langle g, h_K \rangle$. Then by (3),

$$g = \sum_K b_K \sum_{L \ge K} (-1)^{|L|-|K|} F_L$$

$$= \sum_L F_L \sum_{K \le L} (-1)^{|L|-|K|} b_K = \sum_L F_L \langle g, \sum_{K \le L} (-1)^{|L|-|K|} h_K \rangle$$

$$= \sum_L F_L \langle g, S_L \rangle .$$

COROLLARY 4. Suppose that $\Gamma(P)$ is symmetric. Then the number of permutations π in $\mathcal{L}(P)$ with $C(\pi) = L$ is $\langle \Gamma(P), S_L \rangle$.

We have found one class of posets with symmetric generating functions, the posets P_L for which $\Gamma(P_L) = S_L$. A permutation π is in $\mathcal{L}(P_L)$ if

$$\pi^{-1}(i) < \pi^{-1}(i+1) \quad \text{for} \quad i \notin A$$

$$\pi^{-1}(i+1) < \pi^{-1}(i) \quad \text{for} \quad i \in A ,$$

where A is the subset of $[n-1]$ associated with L. In other words, $\pi \in L(P_L)$ iff $C(\pi^{-1}) = L$. Thus we have:

THEOREM 5. The number of permutations π of $[n]$ with $C(\pi^{-1}) = K$ and $C(\pi) = L$ is $\langle S_K, S_L \rangle$.

A formula closely related to Theorem 5 was found by Foulkes [3]. If we expand S_K and S_L in Schur functions and evaluate the scalar product using the orthogonality of Schur functions, we obtain Foulkes's formula.

If $L = (L_1, \cdots, L_k)$, we define $\bar{L}$ to be $(L_k, \cdots, L_1)$. Since $S_L = S_{\bar{L}}$, we have the following result of Foata and Schützenberger [2].

COROLLARY 6. The number of permutations π with $C(\pi^{-1}) = K$ and $C(\pi) = L$ is equal to the number with $C(\pi^{-1}) = K$ and $C(\pi) = \bar{L}$.

The only known posets P for which $\Gamma(P)$ is symmetric are those which correspond to column-strict skew plane partitions. Suppose that $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_\ell)$ are partitions, with $\ell \le k$, and that $\mu_i \le \lambda_i$ for each i, where we take μ_i to be zero for $i > \ell$. Then we define a <u>column-strict plane partition</u> of shape λ/μ to be an array a_{ij} of positive integers defined for $1 \le i \le k$, $\mu_i < j \le \lambda_i$ satisfying

$$a_{ij} \le a_{i,j+1} \quad \text{and} \quad a_{ij} < a_{i+1,j}$$

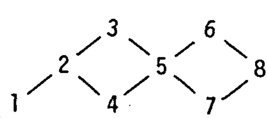
whenever both sides of the inequality are defined. (What we have described is often called a column-strict reverse skew plane partition.) Thus, for example, a column-strict plane partition of shape 443/21 is (in the French notation)

$$\begin{matrix} 3 & 3 & 4 & \\ \cdot & 2 & 3 & 3 \\ \cdot & \cdot & 1 & 2 \end{matrix}$$

It is clear that column-strict plane partitions of shape λ/μ are P-partitions for a certain poset $P = P_{\lambda/\mu}$. To construct $P_{\lambda/\mu}$ we first fill in the shape λ/μ with the numbers from 1 to $n = \Sigma\lambda_i - \Sigma\mu_i$ as follows:

$$\begin{matrix} 1 & 2 & 3 & \\ \cdot & 4 & 5 & 6 \\ \cdot & \cdot & 7 & 8 \end{matrix} \tag{6}$$

Then we obtain $P_{\lambda/\mu}$ by rotating this array 45° counterclockwise:



A <u>Young tableau</u> of shape λ/μ is a plane partition of shape λ/μ in which the parts are $1,2,\cdots,n$. The elements of $L(P_{\lambda/\mu})$ may be identified with Young tableaux of shape λ/μ: given a permutation π in $L(P_{\lambda/\mu})$ we replace i in the array analogous to (6) by $\pi^{-1}(i)$, to obtain a Young tableaux $T(\pi)$. Thus in our example, if $\pi = 74581263$ then $\pi^{-1} = 56823714$ and $T(\pi)$ is

$$\begin{array}{cccc} 5 & 6 & 8 & \\ \cdot & 2 & 3 & 7 \\ \cdot & \cdot & 1 & 4 \end{array}$$

The descent set of π is the set of i in $[n-1]$ for which $i+1$ appears in a higher row than i. Thus in our example; $D(\pi) = \{1,4,7\}$. We define $C(T(\pi))$ to be $C(\pi)$. It is convenient to say that a permutation π is <u>compatible</u> with the shape λ/μ if π is in $L(P_{\lambda/\mu})$.

It can be shown that that $\Gamma(P_{\lambda/\mu})$ is a symmetric function. (For a simple combinatorial proof, see Bender and Knuth [1].) It is called a <u>skew Schur function</u> and denoted $s_{\lambda/\mu}$. Note that S_L is a skew Schur function corresponding to a special kind of shape called a <u>skew hook</u> (also called a <u>rim-hook</u> or <u>border strip</u>). Thus $S_{212} = s_{322/11}$, since f satisfies $f(1) \le f(2) > f(3) > f(4) \le f(5)$ iff

$$\begin{array}{ccc} f(1) & f(2) & \\ \cdot & f(3) & \\ \cdot & f(4) & f(5) \end{array}$$

is a column-strict plane partition of shape 322/11.

Then from Corollary 4 we have:

THEOREM 7. $\langle s_{\lambda/\mu}, S_L\rangle$ is the number of skew Young tableaux T of shape λ/μ with $C(T) = L$, or equivalently, the number of permutations π compatible with the shape λ/μ, with $C(\pi) = L$.

We note that Theorem 7 can be derived from a result of Zelevinsky [14] which gives a combinatorial interpretation for the general $\langle s_{\lambda/\mu}, s_{\alpha/\beta}\rangle$.

3. MULTIPARTITE P-PARTITIONS. Let $X_1, X_2,\cdots, X_r$ be totally ordered infinite sets. Then $X^{(r)} = X_1 \times X_2 \times\cdots\times X_r$ may be totally ordered lexicographically. (Thus if $X_1 = X_2 = \mathbb{N}$, we have $(0,0) < (0,1) < (1,0) < (1,1)$.) An <u>r-partite P-partition</u> is a P-partition with respect to the totally ordered set $X^{(r)}$.

If f is a function from $[n]$ to $X^{(r)}$ we write $f(i) = (f_1(i), f_2(i),\cdots, f_r(i))$. We write $A_r(P)$ for the set of r-partite P-partitions.

As an example, take $r = 2$ and suppose that P consists of the two incomparable points 1 and 2. By Theorem 1, $A_2(P) = A_2(12) \cup A_2(21)$. Now $A_2(12)$ consists of all $f\colon [2] \to X_1 \times X_2$ with $f(1) \le f(2)$; i.e., $(f_1(1), f_2(1)) \le (f_1(2), f_2(2))$ lexicographically. It is easily checked that $A_2(12)$ is the disjoint union of the sets

$$\{f\colon f_1(1) \le f_1(2) \text{ and } f_2(1) \le f_2(2)\}$$

and

$$\{f\colon f_1(1) < f_1(2) \text{ and } f_2(1) > f_2(2)\}.$$

A similar decomposition exists for $A_2(21)$.

We shall show that such a decomposition exists for any P and any r. The *basic idea of the decomposition is due to Gordon [5] and was further developed* by Garsia and Gessel [4]. By Theorem 1,

$$A_r(P) = \bigcup_{\pi \in L(P)} A_r(\pi)$$

so we need only decompose $A_r(\pi)$. We shall see that the general case follows *fairly easily from the case* $r = 2$.

LEMMA 8. Let g be a function from $[n]$ to the totally ordered set X. Then the following are equivalent:

(a) $g \in A(\sigma)$.

(b) If $i < j$ then $g(\sigma(i)) \le g(\sigma(j))$. If in addition $\sigma(i) > \sigma(j)$, then $g(\sigma(i)) < g(\sigma(j))$.

(c) If $g(i) < g(j)$, or if $i < j$ and $g(i) = g(j)$, then $\sigma^{-1}(i) < \sigma^{-1}(j)$.

PROOF. The defining condition for $A(\sigma)$ is obtained by replacing i and j in (b) by $\sigma^{-1}(i)$ and $\sigma^{-1}(j)$. Thus (a) and (b) are equivalent. Condition (c) is easily seen to be equivalent to the contrapositive of (b).

For any function g defined on $[n]$ and any σ in S_n, we define the function g^σ by $g^\sigma(i) = g(\sigma i)$.

THEOREM 9. Let f be a function from $[n]$ to $X_1 \times X_2$. Then f is in $A_2(\pi)$ iff for some σ in S_n, f_2 is in $A(\sigma)$ and f_1^σ is in $A(\sigma^{-1}\pi)$. Moreover, σ is unique.

PROOF. First suppose $f \in A_2(\pi)$. There is a unique σ such that $f_2 \in A(\sigma)$. *We must show that* $f_1^\sigma \in A(\sigma^{-1}\pi)$, i.e., that

(i) $f_1(\pi(i)) \le f_1(\pi(i+1))$, and

(ii) If $f_1(\pi(i)) = f_1(\pi(i+1)$ then $\sigma^{-1}\pi(i) < \sigma^{-1}\pi(i+1)$.

Since $f \in A_2(\pi)$, we have (i) together with

(iii) If $f_1(\pi(i)) = f_1(\pi(i+1))$ then $f_2(\pi(i)) \le f_2(\pi(i+1))$.

(iv) If $f_1(\pi(i)) = f_1(\pi(i+1))$ and $\pi(i) > \pi(i+1)$ then $f_2(\pi(i)) < f_2(\pi(i+1))$.

Now suppose that $f_1(\pi(i)) = f_1(\pi(i+1))$. Then by (iii) and (iv), either $f_2(\pi(i)) < f_2(\pi(i+1))$ or $f_2(\pi(i)) = f_2(\pi(i+1))$ and $\pi(i) < \pi(i+1)$. Then since $f_2 \in A(\sigma)$, condition (c) of Lemma 8 gives $\sigma^{-1}\pi(i) < \sigma^{-1}\pi(i+1)$, and thus (ii) holds.

Conversely, suppose that $f_2 \in A(\sigma)$ and that $f_1^{\sigma} \in A(\sigma^{-1}\pi)$. We must show that (i), (iii), and (iv) hold. Since $f_1^{\sigma} \in A(\sigma^{-1}\pi)$, (i) is immediate. Now let us suppose that $f_1(\pi(i)) = f_1(\pi(i+1))$. Then $f_1^{\sigma}(\sigma^{-1}\pi(i)) = f_1^{\sigma}(\sigma^{-1}\pi(i+1))$, so since $f_1^{\sigma} \in A(\sigma^{-1}\pi)$, we have $\sigma^{-1}\pi(i) < \sigma^{-1}\pi(i+1)$. Thus since $f_2 \in A(\sigma)$, we have $f_2(\pi(i)) \le f_2(\pi(i+1))$, which is (iii), and by (b) of Lemma 8, if $\pi(i) < \pi(i+1)$ then $f_2(\pi(i)) < f_2(\pi(i+1))$, which is (iv).

Theorem 9 may be restated as follows:

COROLLARY 10. $A_2(\pi)$ is the disjoint union of the sets $A(\sigma_1,\sigma_2)$ over all (σ_1,σ_2) such that $\sigma_2\sigma_1 = \pi$, where $A(\sigma_1,\sigma_2)$ is defined by

$$A(\sigma_1,\sigma_2) = \{f\colon [n] \to X_1 \times X_2 \mid f_1^{\sigma_2} \in A(\sigma_1) \text{ and } f_2 \in A(\sigma_2)\}\ .$$

We now discuss the consequences of Corollary 10 for generating functions. For simplicity we write X and Y for the totally ordered sets X_1 and X_2. We define the weight of (x,y) in $X \times Y$ to be $w(x)w(y)$ where the weights of the elements of X and Y are independent indeterminates. As before, we identify x with $w(x)$, y with $w(y)$, and (x,y) with $w(x,y) = xy$. (We assume that X and Y are disjoint.)

Then we may define $\Gamma_2(P)$ to be the sum of the weights of all bipartite P-partitions. If $q = q(X)$ is any quasisymmetric power series in the variables in X, then $q(XY)$ is well defined in the variables $(x,y) = xy$ for x in X and y in Y. It is then clear that if $\Gamma(P) = \Gamma(P,X)$, then $\Gamma_2(P) = \Gamma(P,XY)$. Applying Corollary 10, we have:

THEOREM 11. $\Gamma_2(\pi) = \Gamma(\pi,XY) = \sum_{\tau\sigma=\pi} F_{C(\sigma)}(X)\, F_{C(\tau)}(Y)$.

It follows from Theorem 11 that the number of pairs (τ,σ) of permutations with $C(\tau) = K$, $C(\sigma) = L$, and $\tau\sigma = \pi$ depends only on $C(\pi)$. As a consequence, we have the following result of Solomon [12]:

COROLLARY 12. In the group algebra of S_n, let g_L be the sum of all π in S_n with $C(\pi) = L$. Then the linear span of the g_L over all compositions L of n is a subalgebra.

Solomon gave an analog of Corollary 12 for all Coxeter groups.

From Theorem 1 and Theorem 11 we obtain:

THEOREM 13. $\Gamma_2(P) = \Gamma(P,XY) = \sum_{\tau\sigma\in L(P)} F_{c(\sigma)}(X)\, F_{c(\tau)}(Y).$

We now look at the special case of Theorem 13 in which $\Gamma(P)$ is symmetric. Here $\Gamma(P,XY)$ is symmetric in the terms xy, so the total ordering is irrelevant. Thus if $a(X)$ is any symmetric function, our definition of $a(XY)$ agrees with the standard "lambda-ring" definition. (See Knutson [8].)

If we expand $s_{\lambda/\mu}(XY)$ as a linear combination of the $F_K(X)\, F_L(Y)$, then the coefficient of $F_K(X)\, F_L(Y)$ is the number of pairs (π,σ) of permutations such that $\sigma\pi$ is compatible with λ/μ, $C(\pi) = K$, and $C(\sigma) = L$. Since $s_{\lambda/\mu}(XY)$ is symmetric in X and Y, we could require instead that $\pi\sigma$ be compatible with λ/μ.

We can extend the scalar product on symmetric functions to symmetric functions in two sets of variables by setting

$$\langle a(X)b(Y),\, c(X)d(Y)\rangle = \langle a(X),c(X)\rangle\, \langle b(X),d(X)\rangle ,$$

and extending by linearity. Then from Theorem 11 we obtain:

THEOREM 14. The number of pairs (π,σ) of permutations such that $\pi\sigma$ is compatible with λ/μ, $C(\pi) = K$, and $C(\sigma) = L$ is $\langle s_{\lambda/\mu}(XY)\, ,\, S_K(X)\, S_L(Y)\rangle$.

There is a multiplication $*$ defined on Sym_n called the <u>inner</u> or <u>internal</u> product which corresponds to the multiplication of characters of the symmetric group S_n. It may be characterized by the property that for any symmetric functions a, b, and c,

$$\langle a(XY),\, b(X)c(Y)\rangle = \langle a(X), b(X)*c(X)\rangle .$$

(To see that this characterization is equivalent to the usual definition, as given by Knutson [8], for example, one can verify it for power sum symmetric functions.) Then Theorem 14 may be restated as follows:

COROLLARY 15. $\langle s_{\lambda/\mu}, S_K * S_L\rangle$ is the number of pairs (π,σ) of permutations such that $\pi\sigma$ is compatible with λ/μ, $C(\pi) = K$, and $C(\sigma) = L$.

A similar (but not obviously equivalent) interpretation to $\langle s_{\lambda/\mu}, S_K * S_L\rangle$ was given by Lascoux [9] in the case in which $\mu = \phi$ and K and L are of the form $(1,\cdots, 1, j)$, so that $s_{\lambda/\mu}$, S_K, and S_L are Schur functions.

We now generalize Corollary 10 to r-partite partitions.

THEOREM 16. $A_r(\pi)$ is the disjoint union of the sets $A(\sigma_1,\sigma_2,\cdots,\sigma_r)$ over all $(\sigma_1,\cdots,\sigma_r)$ such that $\sigma_r\sigma_{r-1}\cdots\sigma_1 = \pi$, where $A(\sigma_1,\cdots,\sigma_r)$ is the set of functions $f\colon [n] \to X_1 \times\cdots\times X_r$ satisfying

$$f_1^{\sigma_r\sigma_{r-1}\cdots\sigma_2} \in A(\sigma_1)$$

$$\cdots$$

$$f_{r-2}^{\sigma_r\sigma_{r-1}} \in A(\sigma_{r-2})$$

$$f_{r-1}^{\sigma_r} \in A(\sigma_{r-1})$$

$$f_r \in A(\sigma_r) \quad .$$

PROOF. We proceed by induction on r. We assume that $r \geq 2$. If f is a function from $[n]$ to $X_1 \times\cdots\times X_r$, let us define $\bar{f}\colon [n] \to X_1 \times\cdots\times X_{r-1}$ by $\bar{f}(i) = (f_1(i),\cdots, f_{r-1}(i))$. Then since $X_1 \times\cdots\times X_r = (X_1 \times\cdots\times X_{r-1}) \times X_r$, by Corollary 10, $A_r(\pi)$ is the disjoint union of the sets $B(\sigma,\sigma_r)$ over all (σ,σ_r) with $\sigma_r\sigma = \pi$, where

$$B(\sigma,\sigma_r) = \{f\colon [n] \to X_1 \times\cdots\times X_r \mid \bar{f}^{\sigma_r} \in A_{r-1}(\sigma) \text{ and } f_r \in A(\sigma_r)\} .$$

If $r \geq 3$, we apply induction to $A_{r-1}(\sigma)$.

In the same way that we derived Corollary 15 from Corollary 10, from Theorem 16 we derive the following:

THEOREM 17. $\langle s_{\lambda/\mu}, S_{L_1} * S_{L_2} * \cdots * S_{L_r} \rangle$ is the number of r-tuples $(\sigma_1,\sigma_2,\cdots,\sigma_r)$ of permutations such that $\sigma_1\sigma_2\cdots\sigma_r$ is compatible with λ/μ and $C(\sigma_i) = L_i$ for $1 \leq i \leq r$.

4. ADDITIONAL REMARKS. We have been concerned here with the descent sets of permutations. However, generating functions for permutations by their number of descents and greater index can easily be obtained from our results. Suppose that the elements of the totally ordered set X are $x_0 < x_1 < x_2 < \cdots$. For $m > 0$ we define a homomorphism Λ_m from formal power series in X to formal power series in q by setting $\Lambda_m(x_i) = q^i$ for $0 \leq i < m$ and $\Lambda_m(x_i) = 0$ for $i > m$. For any composition $L = (L_1,\cdots, L_k)$ of $n > 0$, we define $d(L) = k$ and $I(L) = L_2 + 2L_3 +\cdots+ (k-1)L_k$. Thus if L corresponds to the subset $A = \{a_1,\cdots, a_{k-1}\}$ of $[n-1]$ then $d(L) = |A| + 1$ and $\sum_{i=1}^{k-1} (n-a_i)$. So if $C(\pi) = L$, $d(L)$ is one more than the number of descents of π and $I(L)$ is the "reversed" greater index of π. It is easily verified that if L is a composition of n,

$$\sum_{m=1}^{\infty} t^m \Lambda_m(F_L) = t^{d(L)} q^{I(L)} / (1-t)(1-tq)\cdots(1-tq^n);$$

thus for any poset P on $[n]$,

$$\sum_{m=1}^{\infty} t^m \Lambda_m(\Gamma(P)) = \sum_{\pi\in L(P)} t^{d(\pi)} q^{I(\pi)} / (1-t)(1-tq)\cdots(1-tq^n),$$

where $d(\pi) = d(C(\pi))$ and $I(\pi) = I(C(\pi))$, *and similarly for multipartite* P-partitions.

If B is a set of permutations such that $\sum_{\pi\in B} F_{C(B)}$ is *symmetric then a* formula analogous to Theorem 17 counts r-tuples of permutations with product in B. It is not hard to show that the set of permutations with a given number of inversions has this property. It is also true, but harder to prove, that the set of permutations with a given cycle structure also has this property.

It seems likely that much of the theory developed here generalizes to Coxeter groups other than the symmetric groups. Instead of studying inequalities of the form $f(i) \le f(j)$ we study inequalities determined by the reflecting hyperplanes of the Coxeter group. Thus for the hyperoctahedral groups we consider inequalities of the form $f(i) - f(j) \ge 0$, $f(i) + f(j) \ge 0$, and $f(i) \ge 0$.

Some of the results we have obtained can be expressed most elegantly in the language of coalgebras. (See [6] for definitions and basic properties.) We recall that a coalgebra is a vector space V together with a comultiplication map $\Delta: V \to V \otimes V$ satisfying certain conditions. Any coalgebra V has a dual algebra, which is the dual of V as a vector space, with a multiplication defined by convolution.

First we define a coalgebra S_n^* with basis S_n and comultiplication given by

$$\Delta\pi = \sum_{\tau\sigma=\pi} \sigma \otimes \tau .$$

Next we define another coalgebra $Qsym_n^*$ on the quasisymmetric power series of degree n, where if $F_L(XY) = \sum_{J,K} c_{J,K}^{L} F_J(X) F_K(Y)$, then

$$\Delta F_L = \sum_{J,K} c_{J,K}^{L} F_J \otimes F_K .$$

Theorem 11 asserts that the map $\pi \to F_C(\pi)$ extends by linearity to a (surjective) coalgebra homomorphism $\varphi^*: S_n^* \to Qsym_n^*$. The dual of S_n^* is just the group algebra of S_n and the dual of $Qsym_n^*$ may be identified with $Qsym_n$ with a multiplication defined by

$$F_J * F_K = \sum_L C^L_{J,K} F_L \quad .$$

The dual of φ^* is an injective homomorphism φ from $Qsym_n$ to the group algebra of S_n given by $\varphi(F_L) = \sum_{C(\pi)=L} \pi$.

The comultiplication on $Qsym^*_n$ restricts to symmetric functions to give a coalgebra Sym^*_n . The dual of Sym^*_n is the algebra Sym_n of symmetric functions with the inner product, which is isomorphic to the algebra of generalized characters, or class functions, on S_n. The dual of the injection $Sym^*_n \to Qsym^*_n$ is a surjection θ: $Qsym_n \to Sym_n$ in which $\theta(F_L) = S_L$. Some properties of θ have been given by Solomon [12].

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