A self paired Hopf algebra on double posets and a Littlewood–Richardson rule ☆

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ABSTRACT

Let **D** be the set of isomorphism types of finite double partially ordered sets, that is sets endowed with two partial orders. On $\mathbb{Z}\mathbf{D}$ we define a product and a coproduct, together with an internal product, that is, degree-preserving. With these operations $\mathbb{Z}\mathbf{D}$ is a Hopf algebra. We define a symmetric bilinear form on this Hopf algebra: it counts the number of pictures (in the sense of Zelevinsky) between two double posets. This form is a Hopf pairing, which means that product and coproduct are adjoint each to another. The product and coproduct correspond respectively to disjoint union of posets and to a natural decomposition of a poset into order ideals. Restricting to special double posets (meaning that the second order is total), we obtain a notion equivalent to Stanley's labelled posets, and a Hopf subalgebra already considered by Blessenohl and Schocker. The mapping which maps each double poset onto the sum of the linear extensions of its first order, identified via its second (total) order with permutations, is a Hopf algebra homomorphism, which is isometric and preserves the internal product, onto the Hopf algebra of permutations, previously considered by the two authors. Finally, the scalar product between any special double poset and double posets naturally associated to integer partitions is described by an extension of the Littlewood–Richardson rule.

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1. Introduction

We define a combinatorial Hopf algebra based on *double posets*, endowed with a symmetric bilinear form based on *pictures* between double posets, in analogy to pictures of tableaux as defined by Zelevinsky in [22]. Zelevinsky's definition extends straightforwardly to double posets. This form is a Hopf pairing on the previous Hopf algebra. Thus we obtain a link between two articles of Zelevinsky: the one mentioned above and the contemporary article [23], where he studied self-dual Hopf algebras arising from the character theory of classical finite groups. The results we prove show that pictures are fundamentally linked to scalar products, a point of view already present in Zelevinsky's work, who proved that the scalar product of two skew Schur functions is equal to the number of pictures between their shapes.

Note that pictures had been introduced previously by James and Peel in [12, pp. 351–352]. See also [7] and [13] for the study of pictures between skew shapes.

We call *double poset* a set which is endowed with two partial orders $<_1$ and $<_2$. We consider isomorphism classes of double posets: on the $\mathbb{Z}$-module with basis the set of (isomorphism classes of) double posets, we define combinatorially a product and a coproduct, which will make it a graded Hopf algebra. We define a symmetric bilinear form $\langle x, y \rangle$ defined as the number of pictures from x to y; this is proved to be a Hopf pairing: in other words, product and coproduct are adjoint each of another, that is, one has the following formula:

$$\langle xy, z \rangle = \langle x \otimes y, \delta(z) \rangle.$$

This formula means, roughly speaking, that computing with the product is equivalent to computing with the coproduct. Recall that self-dual Hopf algebras (where the pairing is moreover nondegenerate) play a great role in representation theory, see [9,23]. Here, the form is degenerate. An important example of such a situation (a degenerate Hopf pairing) is the *coplactic bialgebra* of [5]; although degenerate, the pairing is very useful; it is one of the main ingredient in the noncommutative theory of the symmetric group of [5].

When the second order of a double poset is total, one obtains the notion which we call special double poset; it is equivalent to that of labelled poset of Stanley [20], or that of shape of Blessenohl and Schocker [5]. The corresponding submodule is then a sub-bialgebra; this bialgebra has already been considered by Blessenohl and Laue, see [5, pp. 41–42]. We prove that there is a natural homomorphism into the bialgebra of permutations of [15]. This mapping is implicit in Stanley's work (see also [10]). The fact that one has a sub-bialgebra and a homomorphism is already due to [5] (see also [18] and [11]). We give further properties of this homomorphism: it is an isometry, and preserves the internal product (this product extends the product of permutations; it is defined in Section 2.3).

To each integer partition is naturally associated a special double poset; this construction is described in [10]. In Theorem 3.4, we describe the pairing between such a double poset and any special double poset by a rule which extends the Littlewood–Richardson rule.

As mentioned by the referee, related work may be found in [2]. There appears the notion of *Hopf monoid*, which allows to construct in one shot many combinatorial Hopf algebras. This could have been made here, avoiding some verifications of Hopf algebra axioms. We refer the reader specially to Sections 13.1, 13.1.4, 17.4.1 of [2].

Note that all the bialgebras in this article are $\mathbb{Z}$-algebras, are graded and connected (that is, the 0-component is $\mathbb{Z}$), hence these bialgebras are Hopf algebras.

2. The bialgebra on double posets

2.1. The self-dual bialgebra on double posets

A *double poset* is a triple

$$(E, <_1, <_2),$$

where E is a *finite set*, and $<_1$ and $<_2$ are two partial orders on E. This notion was implicit in [16]. When no confusion arises, we denote $(E, <_1, <_2)$ simply by E. We call $<_1$ the *first order* of E and $<_2$ *the second order* of E.

As expected, we say that two double posets $(E, <_1, <_2)$ and $(F, <_1, <_2)$ are *isomorphic* if there exists a bijection $\phi : E \to F$ which is an isomorphism from the partial order $(E, <_1)$ to $(F, <_1)$ and from $(E, <_2)$ to $(F, <_2)$, i.e.

$$\forall x, y \in E: \quad x <_i y \text{ in } E \quad \Leftrightarrow \quad \phi(x) <_i \phi(y) \text{ in } F, \quad \text{for } i = 1, 2.$$

Rather than on double posets, we want to work on isomorphism classes of double posets: to avoid too much notation, we simply say double poset, meaning its isomorphism class.

Let **D** denote the set of double posets. We define some combinatorial operations on this set, which will serve to define the bialgebra structure on $\mathbb{Z}\mathbf{D}$, the set of $\mathbb{Z}$-linear combinations of double posets.

If E and F are two double posets, their *composition*, denoted EF, is the double poset $(E \cup F, <_1, <_2)$, where the union is disjoint and where

- the first order $<_1$ of EF is the extension to $E \cup F$ of the first orders $<_1$ of E and F, and no element of E is comparable to any element of F;
- the second order $<_2$ of EF is the extension to $E \cup F$ of the second orders $<_2$ of E and F, together with the new relations $e <_2 f$ for any $e \in E$, $f \in F$.

The product on $\mathbb{Z}\mathbf{D}$ is obtained by extending linearly the composition on **D**.

Recall that an *inferior ideal* (also called simply *order ideal*) of a poset $(E, <)$ is a subset $I \subseteq E$ such that if $y \in I$ and $x < y$, then $x \in I$. A *superior ideal* (also called *dual order ideal*) of E is a subset $S \subseteq E$ such that if $x \in J$ and $x < y$, then $y \in S$. Clearly, the complement of an inferior ideal is a superior ideal and conversely. A *decomposition* of a poset $(E, <)$ is a couple of posets (I, S) where I is an inferior ideal and S its complement, with their induced order.

We call *decomposition* of a double poset $(E, <_1, <_2)$ a pair

$$\big((I, <_1, <_2), (S, <_1, <_2) \big),$$

where (I, S) is a decomposition of the poset $(E, <_1)$, and where the first and second orders $<_1, <_2$ for I and S are obtained by restricting the orders $<_1, <_2$ of the double poset E.

Now let $\delta : \mathbb{Z}\mathbf{D} \to \mathbb{Z}\mathbf{D} \otimes \mathbb{Z}\mathbf{D}$ be the linear map defined on **D** by

$$\delta \big((E, <_1, <_2) \big) = \sum (I, <_1, <_2) \otimes (S, <_1, <_2), \tag{1}$$

where the sum is extended to all decompositions (I, S) of $(E, <_1, <_2)$.

A *picture* between double posets $(E, <_1, <_2)$ and $(F, <_1, <_2)$ is a bijection $\phi : E \to F$ such that:

- $e <_1 e' \Rightarrow \phi(e) <_2 \phi(e')$ and
- $f <_1 f' \Rightarrow \phi^{-1}(f) <_2 \phi^{-1}(f')$.

In other words, a picture is a bijection ϕ of E to F which is increasing from the first order of E to the second order of F and such that its inverse ϕ^{-1} is increasing from the first order of F to the second order of E. We define a pairing $\langle , \rangle : \mathbb{Z}\mathbf{D} \times \mathbb{Z}\mathbf{D} \to \mathbb{Z}$ for any double posets E, F by:

$$\langle E, F \rangle = \big| \{ \alpha : E \to F, \ \alpha \text{ is a picture} \} \big|,$$

and extend it bilinearly to obtain a symmetric bilinear form on $\mathbb{Z}\mathbf{D}$, which we call *Zelevinsky pairing*.

Theorem 2.1. $\mathbb{Z}\mathbf{D}$ *is a graded Hopf algebra and the form is a Hopf pairing.*

Thus we have for any double posets E, F, G,

$$\langle EF, G \rangle = \langle E \otimes G, \delta G \rangle.$$

Proof of Theorem 2.1. We omit the easy verification of the associativity of the product and of the coassociativity of the coproduct; similarly for the homogeneity of both, where the degree of a double

poset is the number of its elements. In order to show that $\mathbb{Z}\mathbf{D}$ is a bialgebra, we show that the coproduct δ is a homomorphism for the product. It amounts to show that there is a bijection between the set of decompositions of the double poset EF and the set of pairs $(E_i F_i, E_s F_s)$, where (E_i, E_s) is a decomposition of E and (F_i, F_s) is a decomposition of F. The bijection is the natural one: take a decomposition (I, S) of EF; then I is an inferior ideal of $(E \cup F, <_1)$; hence, $I \cap E$ is an inferior ideal of $(E, <_1)$, $I \cap F$ is an inferior ideal of $(F, <_1)$ and $I = (I \cap E)(I \cap F)$. Similarly, $S \cap E$ is a superior ideal of $(E, <_1)$, $S \cap F$ is a superior ideal of $(F, <_1)$ and $S = (S \cap E)(S \cap F)$. Hence the required mapping is the identity mapping (modulo double posets isomorphisms):

$$(I, S) \mapsto \big((I \cap E)(I \cap F), (S \cap E)(S \cap F)\big).$$

To see that it is a bijection, note that if (E_i, E_s) is a decomposition of E and (F_i, F_s) is a decomposition of F, then $E_i \cup F_i$ is an inferior ideal I, and $E_s \cup F_s$ the complementary superior ideal S of $(E \cup F, <_1)$; moreover, $I = E_i F_i$ and $S = E_s F_s$.

We prove now the Hopf pairing property. In view of the identity stated before the proof, this amounts to give, for any double posets E, F, G, a bijection between pictures from EF to G and 4-tuples (ϕ, ψ, I, S), where I is an inferior ideal of G, S the complementary superior ideal, ϕ a picture of E onto I and ψ a picture of F onto S. So, let α be a picture from EF onto G. Define $I = \alpha(E)$, $S = \alpha(F)$ and the bijections $\phi : E \to I$, $\psi : F \to S$ obtained by restriction of α to E and F. We verify first that I is an inferior ideal of G; take g, g' in G with $g <_1 g'$ and $g' \in I$, hence $\alpha^{-1}(g') \in E$. Then, α being a picture, we have $\alpha^{-1}(g) <_2 \alpha^{-1}(g')$. Now, if we had $g \notin I$, then $g \in S$, hence $\alpha^{-1}(g) \in F$, therefore $\alpha^{-1}(g') <_2 \alpha^{-1}(g)$, by definition of the second order of EF: contradiction. Similarly, S is a superior ideal of G. Now, the restriction of a picture is a picture, so ϕ and ψ are pictures.

Conversely, given a 4-tuple as above, we glue together the two bijections ϕ and ψ to obtain a bijection $\alpha : E \to G$. Since in EF elements of E and elements of F are $<_1$-incomparable, the fact that α is increasing from $(EF, <_1)$ onto $(G, <_2)$ follows from the similar property for ϕ and ψ. Now let $g, g' \in G$ with $g <_1 g'$; if they are both in I or both in S, then $\alpha^{-1}(g) <_2 \alpha^{-1}(g')$, by the similar property for ϕ and ψ; otherwise, we have $g \in I$ and $g' \in S$, since I, S are ideals. Then $\alpha^{-1}(g) \in E$ and $\alpha^{-1}(g') \in F$, and consequently, $\alpha^{-1}(g) <_2 \alpha^{-1}(g')$, by the definition of the second order of EF. Thus α is a picture.

The counity map $\epsilon : \mathbb{Z}\mathbf{D} \to \mathbb{Z}$ maps the empty double poset on 1, and all other doubles posets onto 0. It is a morphism for the product. By a well-known fact, a graded connected bialgebra is a Hopf algebra. □

Remark.

1. A bialgebra similar to $\mathbb{Z}\mathbf{D}$ has been defined on posets (not double posets) by Schmitt in [19, Section 16]; see also [1, Example 2.3]. Formally it means that the mapping of $\mathbb{Z}\mathbf{D}$ into the bialgebra of Schmitt, which sends a double poset onto the poset with only the first order $<_1$, is a bialgebra homomorphism.
2. A bialgebra constructed on posets has been considered in [6] and [4], together with a bialgebra homomorphism into quasi-symmetric functions. As mentioned by the referee, the link between their construction and ours can be made by means of the *Birkhoff transform*, see [1, Example 2.4], [2, Section 13.9.1].

2.2. A homomorphism into quasi-symmetric functions

Let $\pi = (E, <_1, <_2)$ be a double poset. Similarly to [20] and [10], we call π-*partition*, a function x from E into a totally ordered set X, such that:

- $e <_1 e'$ implies $x(e) \leqslant x(e')$;
- $e <_1 e'$ and $e \geqslant_2 e'$ implies $x(e) < x(e')$.

Note that if x is injective, the first condition suffices.

Now suppose that X is an infinite totally ordered set of commuting variables. Then the generating quasi-symmetric function of π is the sum, over all π-partitions, of the monomials $\prod_{e \in E} x(e)$. We denote it $\Gamma(\pi)$. By extending linearly Γ to $\mathbb{Z}\mathbf{D}$, we obtain a linear mapping into the algebra of quasi-symmetric functions. For quasi-symmetric functions, see [21, 7.19]. They form a bialgebra denoted **QSym**, see e.g. [15].

Theorem 2.2. $\Gamma : \mathbb{Z}\mathbf{D} \to \mathbf{QSym}$ *is a homomorphism of bialgebras.*

This result is implicit in [17, Proposition 4.6 and Théorème 4.16]. We therefore omit the proof; for the definition of the coproduct of the bialgebra of quasi-symmetric functions, see [10, p. 300], [15, Section 2], [2, Section 17.3].

2.3. An internal product

Let $(E, <_1, <_2)$ and $(F, <_1, <_2)$ be two double posets. Let $\phi : (E, <_1) \to (F, <_2)$ be a bijection and denote its graph by

$$E \times_\phi F = \big\{(e, f): \phi(e) = f\big\}.$$

This set becomes a double poset by: $(e, f) <_1 (e', f')$ if and only if $f <_1 f'$; $(e, f) <_2 (e', f')$ if and only if $e <_2 e'$. In other words, denoting by p_1, p_2 the first and second projections, p_1 is an order isomorphism $(E \times_\phi F, <_2) \to (E, <_2)$ and p_2 is an order isomorphism $(E \times_\phi F, <_1) \to (F, <_1)$. Note that the inverse isomorphisms are $p_1^{-1} = (id, \phi)$ and $p_2^{-1} = (\phi^{-1}, id)$.

Define the *internal product* of $(E, <_1, <_2)$ and $(F, <_1, <_2)$ as the sum of the double posets $E \times_\phi F$ for all increasing bijections $\phi : (E, <_1) \to (F, <_2)$. It is denoted $E \circ F$.

Note that this product has been chosen, among several symmetrical ones, so that the following holds: let σ be a permutation in S_n and denote by P_σ the double poset with underlying set $\{1, \ldots, n\}$, with first order $<_1$ defined by $\sigma(1) <_1 \sigma(2) <_1 \cdots <_1 \sigma(n)$, and with second order $<_2$ as the natural order of this set. Then given two permutations τ and σ, one has

$$P_\sigma \circ P_\tau = P_{\sigma \circ \tau},$$

as the reader may easily verify.

The internal product is compatible with the Zelevinsky pairing, as follows.

Proposition 2.1. *Let E, F, G be double posets. Then $\langle E \circ F, G \rangle = \langle E, F \circ G \rangle$.*

The proposition immediately follows from the following lemma.

Lemma 2.1. *Let $(E, <_1, <_2)$, $(F, <_1, <_2)$, $(G, <_1, <_2)$ be three double posets. There is a natural bijection between*

(i) *the set of pairs (ϕ, α), where ϕ is an increasing bijection $(E, <_1) \to (F, <_2)$ and α is a picture from $E \times_\phi F$ into G;*
(ii) *the set of pairs (ψ, β), where ψ is an increasing bijection $(F, <_1) \to (G, <_2)$ and β a picture from E into $F \times_\psi G$.*

Proof. We show that the bijection is defined by $\psi = \alpha \circ (\phi^{-1}, id)$ and $\beta = (id, \psi) \circ \phi$, the inverse bijection being defined by $\phi = p_1 \circ \beta$, with p_1 the projection $F \times G \to F$ and $\alpha = \psi \circ p_2$ with p_2 the projection $E \times F \to F$.

1. Let (ϕ, α) be as in (i) and define $\psi = \alpha \circ (\phi^{-1}, id)$ and $\beta = (id, \psi) \circ \phi$. Now notice that (ϕ^{-1}, id) is a mapping $F \to E \times_\phi F$ and by definition of the first order of $E \times_\phi F$, it is increasing for the first orders on F and $E \times_\phi F$. Since α is increasing $(E \times_\phi F, <_1) \to (G, <_2)$, we see that ψ is increasing $(F, <_1) \to (G, <_2)$. Now β maps bijectively E into $F \times_\psi G$, as desired, and we verify that it is a picture. Note that $(id, \psi) : F \to F \times_\psi G$ is increasing for the second orders, by

the definition of the latter on $F \times_\psi G$. Hence β is increasing $(E, <_1) \to (F \times_\psi G, <_2)$. Moreover, let $(f, g), (f', g') \in F \times_\psi G$ with $(f, g) <_1 (f', g')$; then $\psi(f) = g$, $\psi(f') = g'$ and $g <_1 g'$. Thus $\psi(f) <_1 \psi(f')$. We have for some $e, e' \in E$, $\beta(e) = (f, g)$ and $\beta(e') = (f', g')$. Note that the definition of ψ implies that $\alpha^{-1} \circ \psi = (\phi^{-1}, id)$, thus $\alpha^{-1}(\psi(f)) = (\phi^{-1}(f), f)$. Since α is a picture, α^{-1} is increasing $(G, <_1) \to (E \times_\phi F, <_2)$, so that $\alpha^{-1}(\psi(f)) <_2 \alpha^{-1}(\psi(f'))$ and therefore by definition of the second order on $E \times_\phi F$, we have $\phi^{-1}(f) <_2 \phi^{-1}(f')$; since $\beta(e) = (\phi(e), \psi(\phi(e)))$, we obtain $f = \phi(e)$ and $e = \phi^{-1}(f)$. Finally $e <_2 e'$ showing that β is a picture.

Define now $\phi' = p_1 \circ \beta$ and $\alpha' = \psi \circ p_2$. We must show that $\phi' = \phi$ and $\alpha' = \alpha$. We have $\phi' = p_1 \circ (id, \psi) \circ \phi$, which is equal to ϕ since $p_1 \circ (id, \psi)$ is the identity of F. Moreover, $\alpha' = \alpha \circ (\phi^{-1}, id) \circ p_2$, and we are done, since $(\phi^{-1}, id) \circ p_2$ is the identity of $E \times_\phi F$.

2. Let (ψ, β) be as in (ii), and define $\phi = p_1 \circ \beta$ and $\alpha = \psi \circ p_2$. Now, β is an increasing bijection $(E, <_1) \to (F \times_\psi G, <_2)$ and p_1 is an increasing bijection $(F \times_\psi G, <_2) \to (F, <_2)$ by definition of the second order on $F \times_\psi G$. Thus ϕ is an increasing bijection $(E, <_1) \to (F, <_2)$. Moreover, p_2 is an increasing bijection $(E \times_\phi F, <_1) \to (F, <_1)$ by definition of the first order of $E \times_\phi F$, and ψ is an increasing bijection $(F, <_1) \to (G, <_2)$. Thus α is an increasing bijection:

$$\alpha : (E \times_\phi F, <_1) \to (G, <_2).$$

We show that α^{-1} is also increasing $(G, <_1) \to (E \times_\phi F, <_2)$. Indeed, let $g, g' \in G$ with $g <_1 g'$. Then $g = \alpha(e, f)$, $g' = \alpha(e', f')$ with $\phi(e) = f$, $\phi(e') = f'$ and we must show that $(e, f) <_2 (e', f')$, that is, $e <_2 e'$. Since β is a picture, β^{-1} is increasing $(F \times_\psi G, <_1) \to (E, <_2)$. We have $g = \alpha(e, f) = \psi(f)$ and similarly $g' = \psi(f')$. Hence $(f, g), (f', g') \in F \times_\psi G$ and $(f, g) <_1 (f', g')$ since $g <_1 g'$. Thus $\beta^{-1}(f, g) <_2 \beta^{-1}(f', g')$. Now $f = \phi(e) = p_1(\beta(e)) \Rightarrow \beta(e) = (f, \psi(f)) = (f, g) \Rightarrow e = \beta^{-1}(f, g)$. Similarly, $e' = \beta^{-1}(f', g')$. It follows that $e <_2 e'$.

Define now $\psi' = \alpha \circ (\phi^{-1}, id)$ and $\beta' = (id, \psi) \circ \phi$. We need to show that $\psi' = \psi$ and $\beta' = \beta'$. We have $\psi' = \psi \circ p_2 \circ (\phi^{-1}, id)$ which is clearly equal to ψ. Moreover, $\beta' = (id, \psi) \circ p_1 \circ \beta$ and we are done since $(id, \psi) \circ p_1$ is the identity of $F \times_\psi G$. □

3. The sub-bialgebra of special double posets

3.1. Special double posets

We call a double poset *special* if its second order is total. Since we identify isomorphic double posets, a special double poset is nothing else than a labelled poset in the sense of [20]. Given a labelled poset, the labelling (which is a bijection of the poset into $\{1, \ldots, n\}$) defines a second order, which is total, on the poset. We denote by $\mathbb{Z}\mathbf{DS}$ the submodule of $\mathbb{Z}\mathbf{D}$ spanned by the special double posets.

A *linear extension* of a special double poset $\pi = (E, <_1, <_2)$ is a total order on E which extends the first order $<_1$ of E. We may identify a total order on E with the word obtained by listing the elements increasingly for this order: let $e_1 e_2 \ldots e_n$ be this word, with $|E| = n$. Let moreover ω be the *labelling* of π, that is the unique order isomorphism from $(E, <_2)$ onto $\{1, \ldots, n\}$. Then we identify the linear extension with the permutation $\sigma = \omega(e_1) \ldots \omega(e_n)$ in S_n; in other words $\sigma(i) = \omega(e_i)$ and the mapping $\sigma^{-1} \circ \omega$ is an increasing bijection $(E, <_1) \to \{1, \ldots, n\}$, since $\sigma^{-1} \circ \omega(e_i) = i$. In this way, a linear extension of π is a permutation $\sigma \in S_n$ such that $\sigma^{-1} \circ \omega$ is an increasing bijection $(E, <_1) \to \{1, \ldots, n\}$.

In [15] a bialgebra structure on $\mathbb{Z}S = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}S_n$ has been constructed. We recall it briefly. Recall that, for any word w of length n on a totally ordered alphabet, the *standard permutation* of w, denoted by $st(w)$, is the permutation which is obtained by giving the numbers $1, \ldots, n$ to the positions of the letters in w, starting with the smallest letter from left to right, then the second smallest, and so on. For example, if $w = 4\,3\,2\,4\,1\,3\,4\,4\,2\,3\,3$, then $st(w) = 8\,4\,2\,9\,1\,5\,10\,11\,3\,6\,7$. The product $\sigma * \tau$ for two permutations $\sigma \in S_n$ and $\tau \in S_p$ is defined as the sum of permutations in S_{n+p} which are in the *shifted shuffle product* of the words σ and τ, that is the shuffle product of σ and $\bar{\tau}$, where the latter word is obtained from τ by replacing in it each digit j by $j + n$. The coproduct δ on $\mathbb{Z}S$ is defined on a permutation $\sigma \in S_n$ by: $\delta(\sigma)$ is the sum, over all factorizations (as concatenation) $\sigma = uv$ of the word σ, of $st(u) \otimes st(v)$, where st denotes standardization of a word. See [15] for these definitions.

On $\mathbb{Z}S$ put the *Jöllenbeck scalar product*, defined by

$$(\sigma, \tau) = \begin{cases} 1 & \text{if } \sigma = \tau^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

It turns $\mathbb{Z}S$ into an self-dual bialgebra, see [5, 5.14] (where actually an isomorphic algebra is considered, obtained by replacing each permutation by its inverse), which means that the scalar product (nondegenerate) is a Hopf pairing.

Define the linear mapping L from $\mathbb{Z}\mathbf{DS}$ into $\mathbb{Z}S$ by sending each special double poset on the sum of its linear extensions. The following result is already in [5, see 4.18, 5.5 and 5.10]. Note that what we call here special double poset is called *shape* by Blessenohl and Schocker. Also, what we call composition is called *semi-direct product* by them. For sake of completeness, we give here an alternative proof of the coalgebra property; we construct a bijection between the appropriate sets of decompositions. The proof in [5] (different from ours) uses an argument, due to Gessel [10], which is inductive on the number of edges in the Hasse diagram of the poset.

Theorem 3.1. $\mathbb{Z}\mathbf{DS}$ *is a sub-bialgebra of* $\mathbb{Z}\mathbf{D}$ *and* $L : \mathbb{Z}\mathbf{DS} \to \mathbb{Z}S$ *is homomorphism of bialgebras.*

Proof. It is straightforward to see that the composition of two special double posets is special, and that a decomposition of a special double poset is a pair of special double posets (the class of special double posets is a *hereditary family* in the sense of Schmitt [19]: it is closed under taking disjoint unions and ideals). Thus, $\mathbb{Z}\mathbf{DS}$ is a sub-bialgebra of $\mathbb{Z}\mathbf{D}$. Moreover, the set of linear extensions of the composition of π and π' is classically the shifted shuffle product of the set of linear extensions of π by that of π'. Hence L is a homomorphism of algebras.

The fact that it is also a homomorphism of coalgebras is proved as follows. Let $\pi = (E, <_1, <_2)$ be a special double poset. Then

$$\delta \circ L(\pi) = \sum_{\sigma, u, v} st(u) \otimes st(v),$$

where the summation is over all triples (σ, u, v), with σ a linear extension of $(E, <_1)$ and where σ is the concatenation uv; moreover,

$$(L \otimes L) \circ \delta(\pi) = \sum_{I, S, \alpha, \beta} \alpha \otimes \beta,$$

where the summation is over all quadruples (I, S, α, β) with I an inferior ideal of $(E, <_1)$ and S its complementary superior ideal, and α, β are respectively linear extensions of I, S for the induced order $<_1$. We show that there is a bijection between the set of such triples and quadruples. To simplify, take $\pi = (E, <_1, <_2)$ with $E = \{1, \ldots, n\}$ and $<_2 \ = \ <$ the natural total order on E. Then the labelling ω of π is the identity mapping. We show that

$$(\sigma, u, v) \mapsto \big(I, S, st(u), st(v)\big),$$

with I the set of naturals appearing in u and S the set of naturals appearing in v, is the desired bijection. Note first that, since σ is a linear extension of $(E, <_1)$ and $\sigma = uv$, then I and S as defined are a lower and a superior ideal of $(E, <_1)$; moreover, $st(u)$ and $st(v)$ are linear extensions of $(I, <_1)$ and $(S, <_1)$. This mapping is injective, since any permutation $\sigma = uv$ is determined by $st(u), st(v)$ and the sets of digits in u and v. We show that it is also surjective: let (I, S, α, β) a quadruple as above, and define uniquely $\sigma = uv$ with $st(u) = \alpha$, $st(v) = \sigma$ and I, S the set of digits in u, v. All we have to show is that σ is a linear extension of $(E, <_1)$. That is: if $e = \sigma(j)$ and $e' = \sigma(k)$, with $e <_1 e'$, then $j < k$. Since σ is the concatenation of u and v, and since $st(u), st(v)$ are linear extensions of I, S for the order $<_1$, this is clear if j, k are both digits in $\{1, \ldots, i\}$ or $\{i+1, \ldots, i+s\}$, with i, s the cardinality of I, S; also, if j is in the first set and k in the second. Suppose by contradiction that j is in the second set and k in the first; then $e \in S$ and $e' \in I$, contradicting the ideal property. □

The homomorphism L has two other properties.

Theorem 3.2. *The homomorphism $L : \mathbb{Z}\mathbf{DS} \to \mathbb{Z}S$ preserves the Zelevinsky pairing and the internal product.*

Recall that the *internal product* of $\mathbb{Z}S$ is simply the product which extends the product on permutations. The next lemma extends known results on classical pictures between skew shapes, cf. [8], [5, Remark 13.6].

Lemma 3.1. *Let π, π' be special double posets. There is a natural bijection between pictures from π into π' and linear extensions of π whose inverse is a linear extension of π'.*

Proof. Let ϕ be a picture from π to π', and denote by ω, ω' the respective labellings. Then, the first condition on a picture means that $\omega' \circ \phi$ is an increasing bijection of $(P, <_1)$ into $\{1, \ldots, n\}$; hence $\omega \circ \phi^{-1} \circ \omega'^{-1}$ is a linear extension of $(P, <_1)$. Similarly, the second condition means that $\omega' \circ \phi \circ \omega^{-1}$ is a linear extension of $(P', <_1)$. Thus the lemma follows. □

Proof of Theorem 3.2. The fact that L preserves the pairing is immediate from the lemma.

It remains to show that L is a homomorphism for the internal product. Let π, π' be two special double posets with underlying sets E, F. We may assume that $E = F = \{1, \ldots, n\}$, and that their second order $<_2$ is the natural order on $\{1, \ldots, n\}$. Then, since the labellings of E and F are the identity mappings, a linear extension of π (resp. π') is a permutation α (resp. β) in S_n such that α^{-1} (resp. β^{-1}) is increasing from $(E, <_1)$ (resp. $(F, <_1)$) into $\{1, \ldots, n\}$.

Now, let ϕ be increasing $(E, <_1) \to (F, <_2) = \{1, \ldots, n\}$. We construct the double poset $\Pi = E \times_\phi F$ as in Section 2.2. Since we identify isomorphic double posets, we may take F as underlying set, with the first order $<_1$ of F as first order of Π, and with second order defined by the labelling $\phi^{-1} : F \to E = \{1, \ldots, n\}$. Then a linear extension σ of Π is a permutation σ such that $\sigma^{-1} \circ \phi^{-1}$ is increasing $(F, <_1) \to \{1, \ldots, n\}$. Define $\alpha = \phi^{-1}$ and $\beta = \phi \circ \sigma$. Then $\alpha^{-1} = \phi$ (resp. $\beta^{-1} = \sigma^{-1} \circ \phi^{-1}$) is increasing $(E, <_1) \to \{1, \ldots, n\}$ (resp. $(F, <_1) \to \{1, \ldots, n\}$), and therefore α and β are linear extensions of π and π' with $\alpha \circ \beta = \sigma$.

Conversely let α and β be linear extensions of π and π'. Put $\sigma = \alpha \circ \beta$ and $\phi = \alpha^{-1}$. Then ϕ is increasing $(E, <_1) \to (F, <_2) = \{1, \ldots, n\}$ and $\sigma^{-1} \circ \phi^{-1} = \beta^{-1}$ is increasing $(F, <_1) \to \{1, \ldots, n\}$. Hence σ is a linear extension of Π.

All this implies that $L(\pi)L(\pi')$, which is the sum of all $\alpha \circ \beta$'s, is equal to $L(\pi \circ \pi')$, which is equal to the sum of all σ's, for all possible ϕ's. □

It is easy to prove also that the submodule of $\mathbb{Z}\mathbf{DS}$ spanned by the *naturally labelled* special double posets, that is, those whose second order is a linear extension of the first order, is a sub-bialgebra of $\mathbb{Z}\mathbf{DS}$ (the class of naturally labelled posets is closed under taking disjoint unions and ideals). It may be possible that one could compute the antipode of this subalgebra, and that of $\mathbb{Z}\mathbf{DS}$, by extending the techniques of Aguiar and Sottile [3], who computed the antipode of $\mathbb{Z}S$ using the weak order on the symmetric group, and a recursive method in posets, as [10, Proof of Theorem 1] and [5, Lemma 4.11].

Recall that for a permutation $\sigma \in S_n$, its *descent composition* $C(\sigma)$ is the composition of n equal to $(c_1, \ldots, c_k)$, if σ viewed as a word has k consecutive ascending runs of length $c_1, \ldots, c_k$ and k is minimum. For example $C(51247836) = (1, 5, 2)$, the ascending runs being $5, 12478, 36$. Recall from [10, p. 291] the definition of the fundamental quasi-symmetric function F_C, for any composition C (see also [21, 7.19] where it is denoted L_α). Then it follows from [15, Theorem 3.3] that the linear function $F : \mathbb{Z}S \to \mathbf{QSym}$ defined by $\sigma \mapsto F_{C(\sigma)}$ is a homomorphism of bialgebras. Recall that the bialgebra homomorphism $\Gamma : \mathbb{Z}\mathbf{D} \to \mathbf{QSym}$ has been defined in Section 3. Then the following result is merely a reformulation of a result of Stanley (see [10, Theorem 1 and Eq. (1), p. 291], or [21, Corollary 7.19.5]).

Corollary 3.3. *The mapping $F \circ L$ is equal to Γ restricted to $\mathbb{Z}\mathbf{DS}$.*

3.2. Littlewood–Richardson rule

A *lattice permutation* (or Yamanouchi word) is a word on the symbols in $\mathbb{P} = \{1, 2, 3, \ldots\}$ such that, for any i, the number of i's in each left factor is not less than the number of $i + 1$'s. For example, 11122132 is such a lattice permutation. Given a word w on the symbols $1, 2, 3, \ldots$, with k the greatest symbol appearing in it, we call *complement* of w, the word obtained from w by exchanging 1 and k in w, then 2 and $k - 1$, and so on. For the word of the above example, its complement is therefore 33322312. The *weight* of a word w is the partition $\nu = 1^{n_1} 2^{n_2} \ldots$, where n_i is the number of i's in w. For the word above, it is the partition $1^4 2^3 3^1$.

Given a special double poset $\pi = (E, <_1, <_2)$ of cardinality n with labelling ω, and a word $a_1 a_2 \ldots a_n$ of length n over a totally ordered alphabet A, we say that w *fits into* π (we take the terminology from [8]) if the function $E \to A$ defined by $e \mapsto a_{\omega(e)}$ is a π-partition. In other words, given a function $f : E \to A$, call *reading word* of f the word $f(\omega(1)) \ldots f(\omega(n))$. Then f is a π-partition if and only if its reading word fits into π. Note the case where the word is a permutation τ: we have that τ fits into π if and only if the word $\tau(1) \ldots \tau(n)$ fits into π. This means that the mapping $e \mapsto \tau(\omega(e))$ is a π-partition, that is, since it is a bijection, is increasing $(E, <_1) \to \{1, \ldots, n\}$.

Given a partition ν of n, we define (as in [10]) a special double poset $\pi_\nu = (E_\nu, <_1, <_2)$ where E_ν is the Ferrers diagram of ν, where $<_1$ is the order induced on E_ν by the natural partial order of $\mathbb{N} \times \mathbb{N}$, and where $<_2$ is given on the elements of E_ν by $(x, y) <_2 (x', y')$ if and only if either $y > y'$, or $y = y'$ and $x < x'$. Recall that there is a well-known bijection between standard Young tableaux of shape ν and lattice permutations of weight ν, see [21, Proposition 7.10.3(d)]. In this bijection, the shape of the tableau is equal to the weight of the lattice permutation.

If π is a special double poset, we denote by $\tilde{\pi}$ the special double poset obtained by replacing the two orders of π by their opposite. Clearly, a permutation σ fits into π (assumed to be special) if and only if $w_0 \circ \sigma \circ w_0$ fits into $\tilde{\pi}$. Recall that $w_0 = n \; n-1 \; \ldots \; 2 \; 1$ denotes the longest element in the group S_n.

Theorem 3.4. *Let π be a special double poset and ν be some partition. Then, the pairing (π, π_ν), that is, the number of pictures from π to π_ν, is equal to:*

 (i) *the number of lattice permutations of weight ν whose complements fit into π;*
(ii) *the number of lattice permutations of weight ν whose mirror images fit into $\tilde{\pi}$.*

Note that part (ii) of this is the classical formulation of the Littlewood–Richardson rule (see [14, (9.2)] or [21, Theorem A1.3.3]), once one realizes that a skew Schur function indexed by a skew shape is equal to the skew Schur function obtained by rotating by 180 degrees that shape (cf. [5, Chapter 11, pp. 109–110]).

We first need the following:

Lemma 3.2. *Let π be a special double poset. A permutation σ is a linear extension of π (with respect to $<_1$) if and only if its inverse fits into π.*

Proof. We may assume that $\pi = (E, <_1, <_2)$ with $E = \{1, \ldots, n\}$ and $<_2$ the natural order of E. Then ω is the identity and therefore a permutation σ is a linear extension of π if and only if σ^{-1} is increasing $(E, <_1) \to \{1, \ldots, n\}$. On the other hand, τ fits into π if and only if it is increasing $(E, <_1) \to \{1, \ldots, n\}$, as noted previously. $\square$

Proposition 3.1. *Let π be a special double poset. A word $w = a_1 \ldots a_n$ fits into π if and only if its standard permutation does.*

This result is equivalent to a result of Stanley, see [10, Theorem 1] or [21, Theorem 7.19.14].

A standard Young tableau of shape ν is the same thing as a π_ν-partition which is a bijection from the Ferrers diagram of ν onto $\{1, \ldots, n\}$. Thus we can speak of the reading word of a tableau, which

is a classical notion. We denote it $read(T)$. We consider also the *mirror reading word* of T, which is the mirror image of $read(T)$. The following result must be well known, but we give a proof for the convenience of the reader. Part (i) is proved in [5, p. 109].

Proposition 3.2. *Let T be a standard Young tableau and w the associated lattice permutation.*

(i) *Let u be the complement of w. Then the reading word of T is equal to the inverse of the standard permutation of u.*
(ii) *Let v be the mirror image of w. Then the mirror reading word of T is equal to the complement of the inverse of the standard permutation of v.*

Consider for instance the Young tableau

$$T = \begin{array}{|c|c|c|c|}
\hline
5 & \multicolumn{3}{c}{} \\
\cline{1-2}
3 & 9 & \multicolumn{2}{c}{} \\
\hline
2 & 6 & 10 & 11 \\
\hline
1 & 4 & 7 & 8 \\
\hline
\end{array}$$

Then its lattice permutation is $w = 1\ 2\ 3\ 1\ 4\ 2\ 1\ 1\ 3\ 2\ 2$. The complement of the latter is $u = 4\ 3\ 2\ 4\ 1\ 3\ 4\ 4\ 2\ 3\ 3$. Standardizing this latter word, we obtain the permutation $8\ 4\ 2\ 9\ 1\ 5\ 10\ 11\ 3\ 6\ 7$, whose inverse is $5\ 3\ 9\ 2\ 6\ 10\ 11\ 1\ 4\ 6\ 7$, which is indeed the reading word of the given tableau, obtained by concatenating its rows, beginning with the last row. This illustrates (i). For (ii), the mirror image of w is $v = 2\ 2\ 3\ 1\ 1\ 2\ 4\ 1\ 3\ 2\ 1$. The standard permutation of v is $5\ 6\ 9\ 1\ 2\ 7\ 11\ 3\ 10\ 8\ 4$. The inverse of this permutation is $4\ 5\ 8\ 11\ 1\ 2\ 6\ 10\ 3\ 9\ 7$. Finally, the complement of this permutation is $8\ 7\ 4\ 1\ 11\ 10\ 6\ 2\ 9\ 3\ 5$, which is indeed the mirror reading word of T.

We need a lemma. For this, we use a variant of the reading word of a tableau. Call *row word* of a standard Young tableau T the permutation, in word form, obtained by reading in increasing order the first row of T, then the second, and so on. Denote it by $row(T)$. For example, the row word of the previous example is

$$row(T) = 1\ 4\ 7\ 8\ 2\ 6\ 10\ 11\ 3\ 9\ 5.$$

Lemma 3.3. *Let w be a word on the alphabet $\mathbb{P}$ of weight equal to the partition $v = (v_1 \geqslant \cdots \geqslant v_k > 0)$ of n. Let γ the longest element in the Young subgroup $S_{v_1} \times \cdots \times S_{v_k}$.*

(i) *Let u be the complement of w. Then $st(u) = w_0 \circ \gamma \circ st(w)$.*
(ii) *Let v be the mirror image of w. Then $st(v) = \gamma \circ st(w) \circ w_0$.*
(iii) *Suppose that w is a lattice permutation and let T be the tableau of shape v corresponding to w. Then $st(w)$ is the inverse of $row(T)$.*

Proof. (i) Note that γ as word is equal to

$$v_1 \ldots 1\ (v_1 + v_2) \ldots (v_1 + 1)\ \ldots\ n \ldots (v_1 + \cdots + v_{k-1} + 1).$$

Hence

$$w_0 \circ \gamma = (v_2 + \cdots + v_k + 1) \ldots n\ (v_3 + \cdots + v_k + 1) \ldots (v_2 + \cdots + v_k)\ \ldots\ 1 \ldots v_k.$$

The proof of (i) then follows by inspection.

(ii) Likewise, one proves that $st(v) \circ w_0 = \gamma \circ st(w)$ by inspection.

(iii) Let $I_1, \ldots, I_k$ denote the successive intervals of $\{1, \ldots, n\}$ of cardinality $v_1, \ldots, v_k$. Let $L_1, \ldots, L_k$ be the set of elements in the successive rows of T. If I, L are two subsets of equal cardinality of $\mathbb{P}$, we denote by $I \nearrow L$ the unique increasing bijection from I into L. We denote also $f_1 \cup \cdots \cup f_k$

the function which restricts to f_i on its domains, assuming the domains are disjoint. Then $row(T)$ is the permutation $\bigcup_{j=1,\dots,k} I_j \nearrow L_j$, and its inverse is therefore $\bigcup_{j=1,\dots,k} L_j \nearrow I_j$. Moreover, the word w is defined by the following condition: for each position $p \in \{1, \dots, n\} = \bigcup_{j=1,\dots,k} L_j$, the p-th letter of w is j if and only if $p \in L_j$. Recall that $st(w)$, viewed as word, is obtained by giving the numbers $1, \dots, n$ to the positions of the letters in w, starting with the 1's from left to right, then 2's, and so on. Therefore $st(w) = \bigcup_{j=1,\dots,k} L_j \nearrow I_j$. $\quad\square$

Proof of Proposition 3.2. (i) We have to prove that $read(T) = st(u)^{-1}$. We know by Lemma 3.3(i) and (iii) that $st(u) = w_0 \circ \gamma \circ st(w)$ and $st(w)^{-1} = row(T)$. Clearly $read(T) = row(T) \circ \delta$, where δ is the permutation

$$(\nu_1 + \cdots + \nu_{k-1} + 1)\dots n \ (\nu_1 + \cdots + \nu_{k-2} + 1)\dots(\nu_1 + \cdots + \nu_{k-1}) \ \dots \ 1\dots\nu_1.$$

Now $\delta = \gamma \circ w_0$. Therefore, $read(T) = row(T) \circ \gamma \circ w_0 = st(w)^{-1} \circ \gamma \circ w_0 = st(u)^{-1}$, since w_0 and γ are involutions.

(ii) We have to show that $read(T) \circ w_0 = w_0 \circ st(v)^{-1}$. We know by Lemma 3.3(ii) that $st(v) = \gamma \circ st(w) \circ w_0$, hence $st(w) = \gamma \circ st(v) \circ w_0$. Using what we have done in (i), we have therefore

$$read(T) = st(w)^{-1} \circ \gamma \circ w_0 = w_0 \circ st(v)^{-1} \circ \gamma \circ \gamma \circ w_0 = w_0 \circ st(v)^{-1} \circ w_0. \quad\square$$

Proof of Theorem 3.4. (i) By Lemma 3.1 and Lemma 3.2, the indicated scalar product is equal to the number of permutations σ which fit into π and whose inverse fits into π_ν. Let H denote the set of complements of lattice permutation of weight ν. By Proposition 3.2(i), the mapping $H \to RWSYT_\nu$, $w \mapsto st(w)^{-1}$ is a bijection, where we denote by $RWSYT_\nu$ the set of reading words of standard Young tableaux of shape ν. By Proposition 3.1, part (i) of the theorem follows.

(ii) Let K denote the set of mirror images of lattice permutations of weight ν. By Proposition 3.2(ii), the mapping $K \to RWSYT_\nu$, $v \mapsto \sigma = w_0 \circ st(v)^{-1} \circ w_0$ is a bijection. Moreover, $\sigma^{-1} = w_0 \circ st(v) \circ w_0$ fits into $\tilde{\pi}$ if and only if $st(v)$ fits into π, that is, by Proposition 3.1, if and only if v fits into π. $\quad\square$

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