Consequences of the Lakshmibai-Sandhya Theorem; the ubiquity of permutation patterns in Schubert calculus and related geometry

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Combinatorics and Geometry

Intro. Schubert calculus $\longrightarrow$ Schubert Geometry

Modern Schubert calculus is the study of effective methods to compute the expansion coefficients of cohomology classes of Schubert varieties

$$[X(u)] \cdot [X(v)] = \sum c_{u,v}^{w} [X(w)]$$

Intersection theory: $c_{u,v}^{w} = \#$ points in $X(u; E_\bullet) \cap X(v; F_\bullet) \cap X(w_0 w; G_\bullet)$.

This is both a combinatorial and a geometrical statement!

Combinatorics and Geometry

For Schubert varieties in Grassmannians, we have tools:

1. Littlewood-Richardson tableaux
2. Yamanouchi words
3. Knutson-Tao puzzles
4. Vakil's toric degenerations

In general, we don't yet have analogs of all these beautiful tools for other types of Schubert varieties.

Goal. Understand both the combinatorics and geometry of Schubert varieties in order to do Schubert calculus for all types of Schubert varieties.

Outline of Lecture 1

Some Classical Results on the Geometry of Schubert varieties

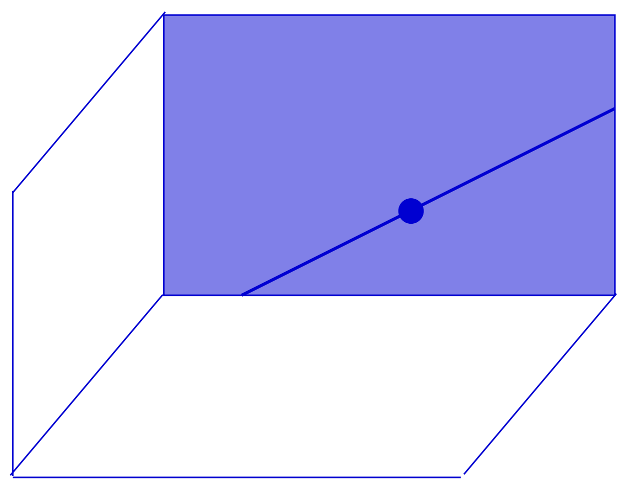
1. Review of Schubert varieties in flag manifolds
2. Tangent space bases
3. Characterizing smooth Schubert varieties

The Flag Manifold

Def. A *complete flag* $F_\bullet = (F_1, \ldots, F_n)$ in $\mathbb{C}^n$ is a nested sequence of vector spaces such that $\dim(F_i) = i$ for $1 \leq i \leq n$. $F_\bullet$ is determined by an ordered basis $\langle f_1, f_2, \ldots f_n \rangle$ where $F_i = \text{span}\langle f_1, \ldots, f_i \rangle$.

Example.

$$F_\bullet = \langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4, \quad e_2 \rangle$$



The Flag Manifold

Canonical Form. Every flag can be represented as a matrix in column echelon form.

$$F_\bullet = \langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4, \quad e_2 \rangle$$

$$\approx \begin{bmatrix} 6 & 4 & 9 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 7 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2, \quad 2e_1 + e_3, \quad 7e_1 + e_4, \quad e_1 \rangle$$

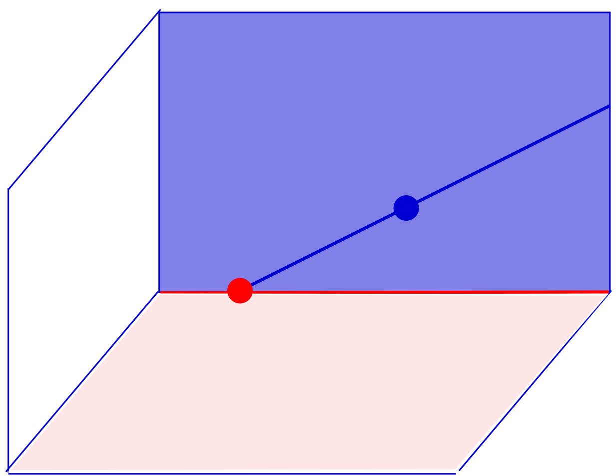
$\mathcal{F}l_n(\mathbb{C}) :=$ *flag manifold* over $\mathbb{C}^n = \{$complete flags $F_\bullet\}$

$= GL_n(\mathbb{C})/B$, where $B =$ upper triangular mats in GL_n

Flags and Permutations

Example. $F_\bullet = \langle 2e_1{+}e_2,\ \ 2e_1{+}e_3,\ \ 7e_1{+}e_4,\ \ e_1\rangle \approx \begin{bmatrix} 2 & 2 & 7 & \textcircled{1} \\ \textcircled{1} & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \end{bmatrix}$

Note. If a flag is written in canonical form, the positions of the leading 1's form a permutation matrix. There are 0's to the right and below each leading 1. This permutation determines the *position* of the flag $F_\bullet$ with respect to the reference flag $E_\bullet = \langle e_1,\ e_2,\ e_3,\ e_4\ \rangle$ drawn in blue.



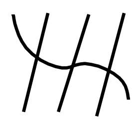
Many ways to represent a permutation

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = \mathbf{2341} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

matrix notation	two-line notation	one-line notation	rank table

$$\begin{matrix} * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \quad = $$

1234



2341

$$= \quad (\mathbf{1}, \mathbf{2}, \mathbf{3})$$

diagram of a permutation	string diagram	reduced word

The Schubert Cell $C_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

Def. $C_w(E_\bullet)$ = All flags $F_\bullet$ with $\mathbf{position}(E_\bullet, F_\bullet) = w$

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) = \mathrm{rk}(w[i,j])\}$$

Example. $$F_\bullet = \begin{bmatrix} 2 & 2 & 7 & ① \\ ① & 0 & 0 & 0 \\ 0 & ① & 0 & 0 \\ 0 & 0 & ① & 0 \end{bmatrix} \in C_{4123} = \left\{ \begin{bmatrix} * & * & * & 1 \\ 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \end{bmatrix} : * \in \mathbb{C} \right\}$$

Easy Observations.

- $\mathtt{dim}_{\mathbb{C}}(C_w) = \#$ *inversions* of $w = \#\{w(i) > w(j) : i < j\} = \ell(w)$
- $C_w = B \cdot w$ is a B-orbit using the left B action, e.g.

$$\begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ 0 & b_{2,2} & b_{2,3} & b_{2,4} \\ 0 & 0 & b_{3,3} & b_{3,4} \\ 0 & 0 & 0 & b_{4,4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} b_{1,2} & b_{1,3} & b_{1,4} & b_{1,1} \\ b_{2,2} & b_{2,3} & b_{2,4} & 0 \\ 0 & b_{3,3} & b_{3,4} & 0 \\ 0 & 0 & b_{4,4} & 0 \end{bmatrix}$$

The Schubert Variety $X_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

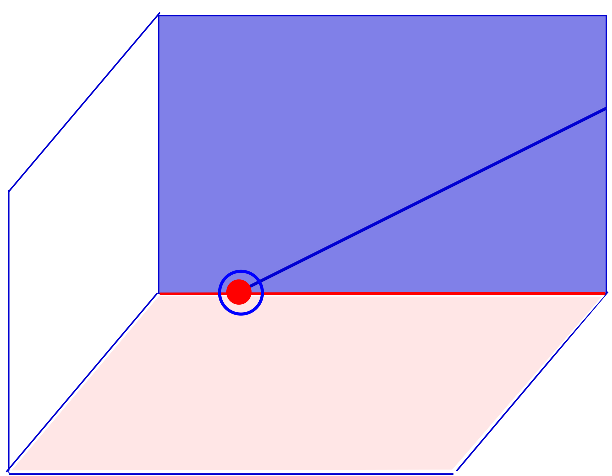
Defn. $X_w(E_\bullet)$ = Closure of $C_w(E_\bullet)$ under the Zariski topology

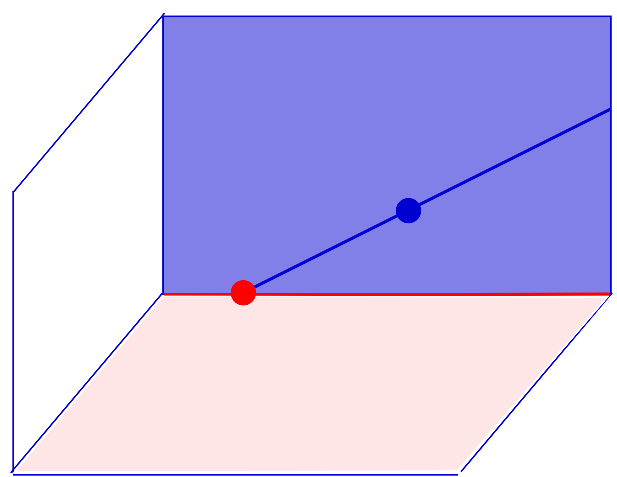
$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) \leq \mathrm{rk}(w[i,j])\}$$

where $E_\bullet = \langle e_1,\ e_2,\ e_3,\ e_4\ \rangle$.

Example. $$\begin{bmatrix} \textcircled{1} & 0 & 0 & 0 \\ 0 & * & * & \textcircled{1} \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \end{bmatrix} \in X_{4123}(E_\bullet) = \overline{\left\{ \begin{bmatrix} * & * & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right\}}$$

Why?. Think about both determinantal equations and pictures.





Combinatorics and Geometry

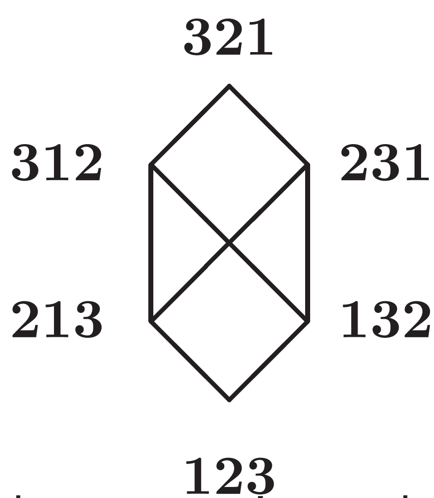
Fact. The closure relation on Schubert varieties defines a nice partial order.

$$X_w = \bigcup_{v \leq w} C_v \quad = \bigcup_{v \leq w} X_v$$

Bruhat order (Ehresmann 1934, Chevalley 1958) is the transitive closure of

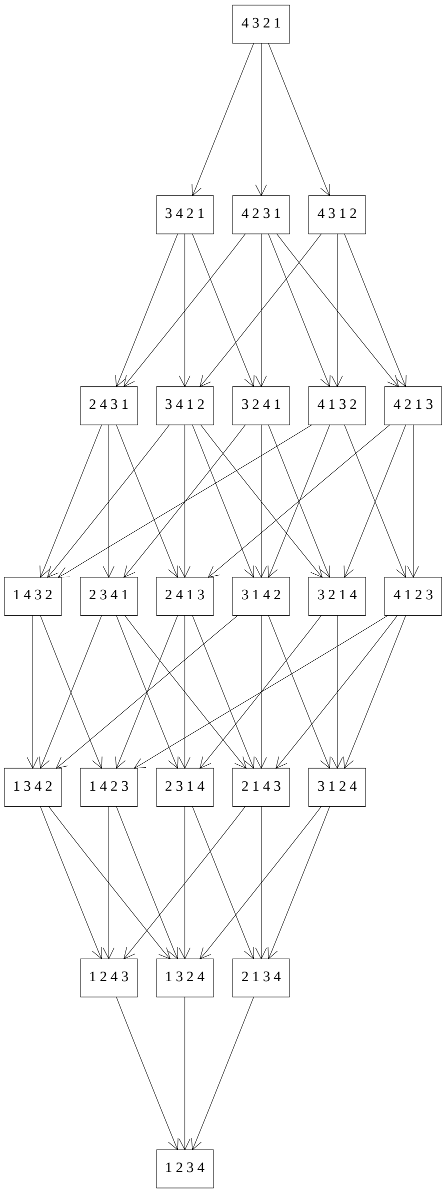
$$w < wt_{ij} \iff w(i) < w(j).$$

Example. Bruhat order on permutations in S_3.

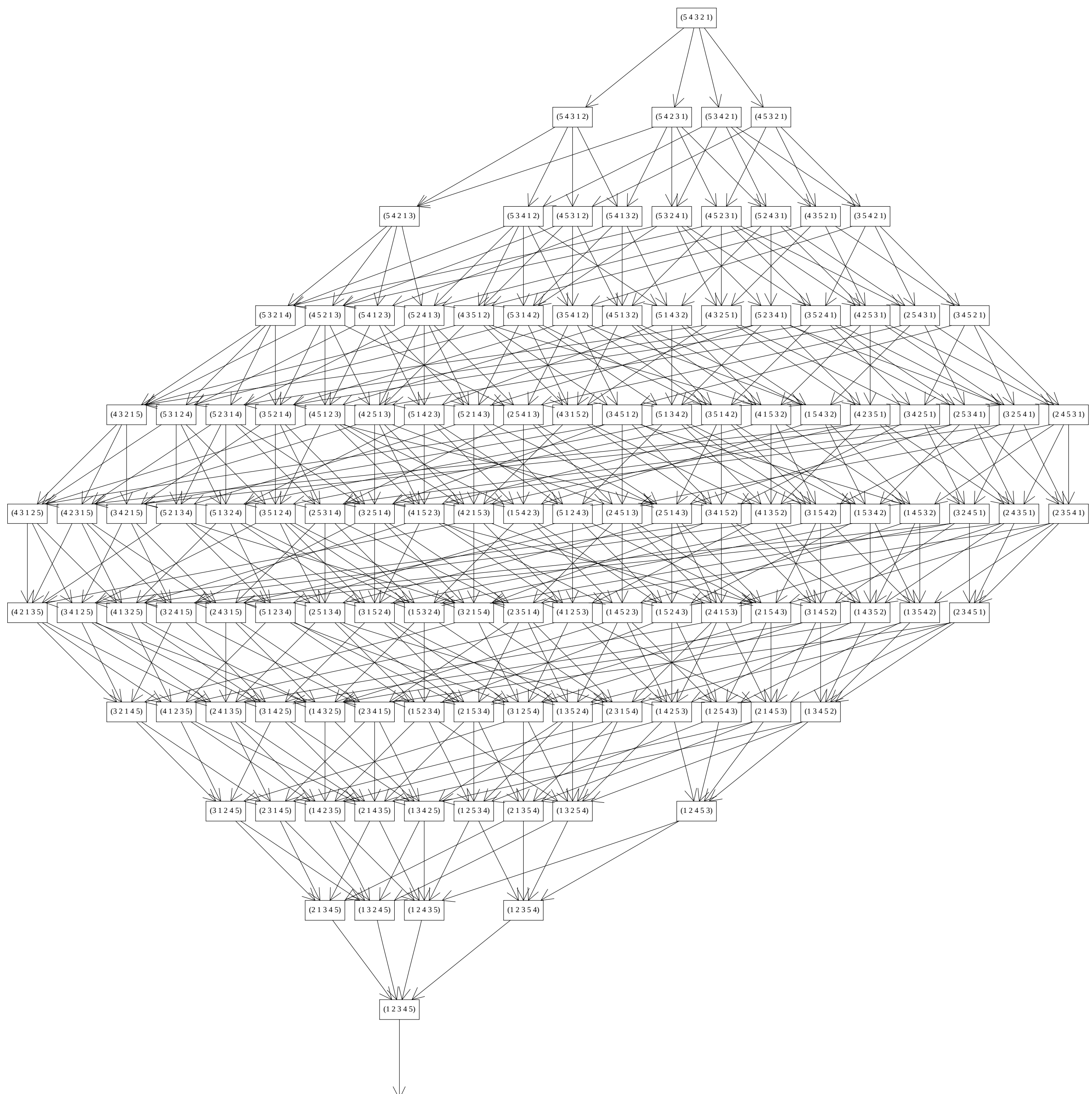


Observations. Self dual, rank symmetric, rank unimodal.

Bruhat order on S_4



Bruhat order on S_5



Poincaré polynomials

Fact. The *Poincaré polynomial* for $H^*(X_w)$ is $P_w(t) = \sum_{v \leq w} t^{l(v)}$.

Example. $w = 3412$

$$\begin{array}{ll} 4: & (3412) \\ 3: & (3142)(3214)(1432)(2413) \\ 2: & (3124)(1342)(2143)(2314)(1423) \\ 1: & (2134)(1243)(1324) \\ 0: & (1234) \end{array}$$

Poincaré polynomial:. $P_{3412}(t) = 1 + 3t + 5t^2 + 4t^3 + t^4$.

10 Fantastic Facts on Bruhat Order

1. Bruhat Order Characterizes Inclusions of Schubert Varieties
2. Contains Young's Lattice in S_∞
3. Nicest Possible Möbius Function: $\mu(v, w) = (-1)^{\ell(w)-\ell(v)}$
4. Beautiful Rank Generating Function: $\prod_{k=1}^{n-1}(1 + t + t^2 + \cdots + t^k)$
5. $[x, y]$ Determines the Composition Series for Verma Modules
6. Symmetric Interval $[\hat{0}, w] \iff X(w)$ rationally smooth
7. Order Complex of (u, v) is shellable
8. Rank Symmetric, Rank Unimodal and k-Sperner
9. Efficient Methods for Comparison
10. Amenable to Pattern Avoidance

Observations or HW Exercises

1. Boundary of X_w has codimension 1.

2. C_w is a dense open set in X_w.

3. Rank conditions of the form $\mathbf{rk}(g) = \mathbf{rk}(w)$ give rise to open sets in Zariski topology using determinantal equations.

4. Rank conditions of the form $\mathbf{rk}(g) \leq \mathbf{rk}(w)$ give rise to closed sets.

5. X_w embeds into projective space $\mathbb{P}^N$ via Plücker coordinates: list all lower left minors.

6. If $w_0 = [n, n-1, \ldots, 1]$, then $GL_n/B = X_{w_0}$.

7. The point w_0 has an affine neighborhood C_{w_0} of dimension $\binom{n}{2}$ and a local coordinate system. The point g has an affine neighborhood $gw_0C_{w_0}$.

8. GL_n acts transitively on the points in the flag manifold so it is a manifold and a projective variety.

9. The flag manifold is *smooth* $=$ non-singular at every point.

Smooth Schubert varieties

Question. Which Schubert varieties are smooth?

Def. A point p on a variety X is *nonsingular* if the dimension of the tangent space to p for X has the same dimension as X. Otherwise p is *singular*.

Jacobian Criterion.: If $I(X) = \langle f_1, \ldots, f_k \rangle$, define the *Jacobian matrix* $J(x) = (\partial f_i / \partial x_j)$. Then, $\mathbf{rk}(J(p)) = \mathbf{codim} X \iff X$ is nonsingular at p.

One Approach. Write down equations for Schubert varieties, compute Jacobians, evaluate at points.

Smooth Schubert varieties

Simplifications.

- A point $p \in C_v \subset X_w$ is singular $\iff$ every point in C_v is singular.
- The set of singular points in any variety is a closed set.
- X_w is smooth $\iff$ X_w smooth at I=identity matrix.
- $Y(w, id) = X_w \cap w_0 C_{w_0}$ = affine neighborhood of X_w containing I.
- Use Fulton's essential set to get a minimal set of required rank conditions.

Smooth Schubert varieties

Example. Is X_{2413} smooth?

$$w_0C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x_{21} & 1 & 0 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & 1 & \end{bmatrix}$$

$Y(2413, I)$ defined by $\langle x_{41}, x_{42}, x_{21}x_{32} - x_{31}1\rangle$.

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ x_{32} & -1 & x_{21} & 0 & 0 \end{bmatrix} \qquad J(I) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

Yes! $\mathrm{rk} J(I) = 3 = \binom{4}{2} - \ell(2413) = \mathrm{codim} X_{2413}$,

Smooth Schubert varieties

Example. Is X_{3412} smooth?

$Y(3412, I)$ defined by $f_1 = x_{41}$ and

$$f_2 = \det \begin{bmatrix} x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \\ x_{41} & x_{42} & x_{43} \end{bmatrix} \equiv x_{21}(x_{32}x_{43} - x_{42}) - x_{31}x_{43}$$

$$J(I) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

No! $\mathrm{rk} J(I) = 1 < 2 = \binom{4}{2} - \ell(3412) = \mathrm{codim} X_{3412}$,

Smooth Schubert varieties

Question. Which Schubert varieties are smooth?

Alternative Approach.

- Think of Schubert varieties as a discrete moduli space with common properties described in some global way.
- Give a description of a basis for the tangent spaces for X_w at $v \leq w$ using Lie algebras.

Lie Algebras

Def. For any subgroup H of GL_n, the *Lie algebra* of H, $\mathbf{Lie}(H)$, is the space of vectors tangent to H at I in $M_{n\times n}$ (= all $n \times n$ matrices).

Def. H is a linear algebraic group if H is a subgroup of GL_n defined by polynomial equations on $M_{n\times n}$.

Useful Facts.

- If H is a linear algebraic group, the connected component of H containing I can be recovered from $\mathbf{Lie}(H)$.
- $\mathbf{Lie}(H) = \{A \in M_{n\times n} : I + A\varepsilon \in H\}$ where $A\varepsilon$ is an infinitesimal vector in the direction of A.

Lie Algebras

Example. $SL_n = V(\det - 1)$ is a linear algebraic group.
What is its Lie algebra?

$$A \in \text{Lie}(H) \iff I + A\varepsilon \in H \iff \det(I + A\varepsilon) = 1$$

Answer.

Lie Algebras

Example. $SL_n = V(\det - 1)$ is a linear algebraic group.
What is its Lie algebra?

$A \in \mathrm{Lie}(SL_n) \iff I + A\varepsilon \in SL_n \iff \det(I + A\varepsilon) = 1 \iff \mathrm{trace}(A) = 0.$

Answer. $\mathrm{Lie}(SL_n) = \{A \in M_{n\times n} : \mathrm{trace}(A) = 0\}$.

Lie Algebras/Tangent space Basis

Example. $\mathfrak{g} = \mathrm{Lie}(SL_n) = \{A \in M_{n\times n} : \mathrm{trace}(A) = 0\}$ has vector space basis $\{E_{i,j}, E_{j,i} : 1 \le i < j \le n\} \cup \{H_i : 1 \le i < n\}$ where

- $E_{i,j}$ is mat with 1 in (i,j)-entry, 0's elsewhere.
- H_i is diagonal mat with $h_{i,i} = 1$, $h_{i+1,i+1} = -1$, 0's elsewhere.

Example. $\mathfrak{b} = \mathrm{Lie}(B \cap SL_n) = \mathrm{span}\{E_{i,j} : i < j\} \cup \{H_i : i < n\}$.

Observation. $GL_n/B = SL_n/(B \cap SL_n)$ so tangent spaces isomorphic.

$$\mathfrak{g}/\mathfrak{b} = \mathrm{span}\{E_{j,i} : i < j\}.$$

Bijection: $\{E_{j,i} : i < j\} \longleftrightarrow \{t_{i,j} : i < j\} = R =$ (reflections)

Lie Algebras/Tangent space Basis

More generally, for any $v \in S_n$, the tangent space to G/B at v is

$$T_v(G/B) = v^{-1}\,(\mathfrak{g}/\mathfrak{b})\,v = \text{span}\{v^{-1}\,E_{j,i}\,v : i < j\}.$$

Why? $G/B \approx G/v^{-1}Bv$, changes the base flag to flag determined by v.

Observations.

- $v^{-1}\,E_{ij}\,v = E_{v(i),v(j)}$
- $t_{v(i),v(j)}\,v = v\,t_{ij}$

Tangent space Bases

Thm. (Lakshmibai-Seshadri) For $v \leq w \in S_n$,

$$T_v(X_w) = \mathrm{span}\{E_{v(j),v(i)} : i < j, \quad vt_{ij} \leq w\},$$

$$\dim T_v(X_w) = \#\{(i < j) : \ vt_{ij} \leq w\}.$$

Proof. $X_w \subset X_{w_0} = G/B \implies T_v(X_w) \subset T_v(X_{w_0)}$
$\implies$ Only need to check which $E_{v(j),v(i)}$ have $I + E_{v(j),v(i)}\varepsilon \in X_w$.

Tangent Space Bases

If $v = id \in S_n$,

$$\mathrm{rk}(I+E_{5,2}\varepsilon) = \mathrm{rk}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & \varepsilon & & & 1 \end{pmatrix} = \mathrm{rk}\begin{pmatrix} 1 & & & & \\ & & & & 1 \\ & & 1 & & \\ & & & 1 & \\ & 1 & & & \end{pmatrix} = \mathrm{rk}(t_{5,2})$$

Only one ε in any submatrix so doesn't effect vanishing of any minor.

Therefore, $(I + E_{j,i}\varepsilon) \in X_w \iff t_{ij} \le w$.

Tangent Space Bases

If $v = id \in S_n$,

$$\mathrm{rk}(I + E_{5,2}\varepsilon) = \mathrm{rk}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & \varepsilon & & & 1 \end{pmatrix} = \mathrm{rk}\begin{pmatrix} 1 & & & & \\ & & & & 1 \\ & & & 1 & \\ & & & & 1 \\ & 1 & & & \end{pmatrix} = \mathrm{rk}(t_{5,2})$$

Only one ε in any submatrix so doesn't effect vanishing of any minor.

Therefore, $(I + E_{j,i}\varepsilon) \in X_w \iff t_{ij} \leq w$.

In general, assume $v < w \in S_n$.

- If $(v + E_{v(j),v(i)}\varepsilon) \in X_v$ then also in X_w, so $E_{v(j),v(i)} \in T_v(X_w)$.
- Otherwise, $v < v_{t_{ij}}$ and $E_{v(j),v(i)} \in T_v(X_w) \iff vt_{ij} \leq w$ by a similar analysis of the rank table for $(v + E_{v(j),v(i)}\varepsilon)$. $\square$

Consequences

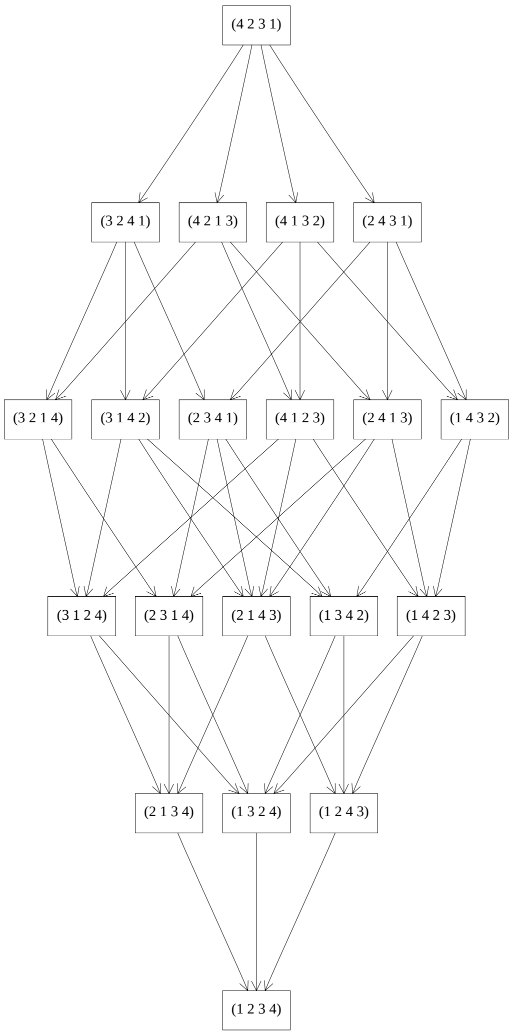
Cor. X_w smooth at v $\iff$ $\mathbf{dim} T_v(X_w) = \#\{t_{ij} : vt_{ij} \leq w\} = \ell(w)$.

$$\iff = \#\{t_{ij} : v < vt_{ij} \leq w\} = \ell(w) - \ell(v).$$

Example. Is X_{4231} smooth?

Answer.

Bruhat Interval $[id, 4231]$



Consequences

Cor. X_w smooth at v $\iff$ $\dim T_v(X_w) = \#\{t_{ij} : vt_{ij} \le w\} = \ell(w)$.

$$\iff = \#\{t_{ij} : v < vt_{ij} \le w\} = \ell(w) - \ell(v).$$

Example. Is X_{4231} smooth?

Answer. No! $v = 2143$ has $6 = 1 + \dim(X_w)$ edges adjacent to it in the Hasse diagram of $\{v \le 4231\}$. Also, $\#\{t_{ij} \le 4231\} = 6 = \dim T_{id}(X_{4231})$.

Consequences

Cor. X_w smooth at $v \iff \dim T_v(X_w) = \#\{t_{ij} : vt_{ij} \leq w\} = \ell(w)$.

$$\iff = \#\{t_{ij} : v < vt_{ij} \leq w\} = \ell(w) - \ell(v).$$

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Singular Locus.

- $\mathrm{Sing}(X_{4231}) = X_{2143}$
- $\mathrm{Sing}(X_{3412}) = X_{1324}$.

All other Schubert varieties X_w for w in S_4 are smooth.

Bruhat graphs

Def. The *Bruhat graph* for w has vertex set $\{v \in S_n : v \leq w\} = [id, w]$ and edges (v, vt_{ij}) if both $v, vt_{ij} \leq w$.

Observe. $\dim T_v(X_w)$ degree of v in the Bruhat graph for w.

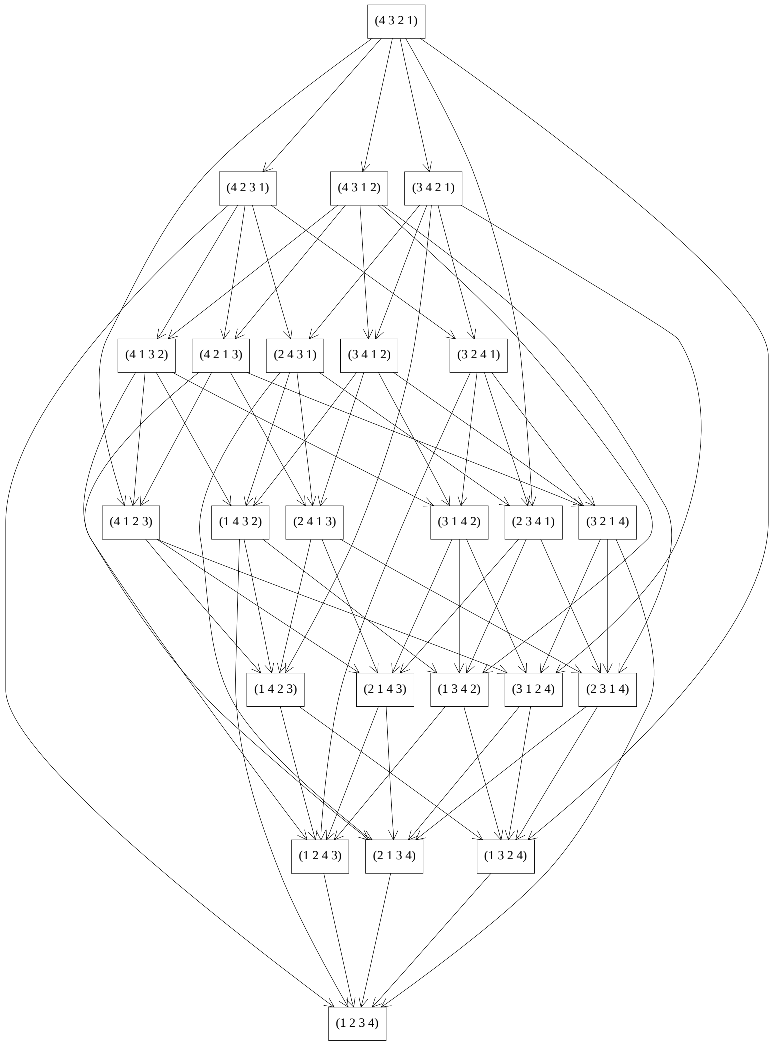
Geometrical Interpretation/Observations.

- T-fixed points: The permutation matrices in GL_n/B are exactly the points fixed under left multiplication by $T =$ the invertible diagonal matrices.

- $T' \subset T$-fixed curves: If $v < vt_{ij}$, set

$$L_v = \{v + * E_{j,v(i)} : * \in \mathbb{C}\} \cup \{vt_{ij}\} \approx \mathbb{P}^1.$$

Bruhat Graph $w = 4321$



Bruhat graphs

GKM Spaces. (Goresky-Kottwitz-MacPherson) Any symplectic manifold with a T=torus action, a discrete set of T-fixed points, and curves connecting these fixed points which are invariant under a codimension 1 subtorus of T.

Questions. What can be said about the manifold using the information in the graph induced by the fixed points and curves?

See work of Baird, Bialynicki-Birula, Chang, Goldin, Goresky, Guillemin, Harada, Harada, Hausmann, Henriques, Holm, Jeffrey, Kirwan, Knutson, Kottwitz, MacPherson, Mare, Matsumura, Sabatini, Sjamaar, Skjelbred, Tolman, Tymoczko, Weitsman, Zara.

Lakshmibai-Sandhya Theorem

Fact. There exists a simple criterion for characterizing smooth Schubert varieties using permutation pattern avoidance. (Knuth, Pratt, Tarjan)

Lakshmibai-Sandhya Theorem

Fact. There exists a simple criterion for characterizing smooth Schubert varieties using permutation pattern avoidance.

Theorem: Lakshmibai-Sandhya 1990 (see also Haiman, Ryan, Wolper)
X_w is non-singular $\iff$ w has no subsequence with the same relative order as 3412 and 4231.

	$w = 625431$	contains	$6241 \sim 4231$	$\Longrightarrow$	X_{625431} is singular
Example:	$w = 612543$	avoids	4231 $\&3412$	$\Longrightarrow$	X_{612543} is non-singular

Lakshmibai-Sandhya Theorem

Proof Overview (in retrospect).

Step 1. If w contains a 3412 or 4231, then X_w has a singular point v obtained from w by rearranging the numbers in the pattern to 1324 or 2143. Use the tangent space basis characterization.

Step 2. If $w \in S_n$ avoids 3412 and 4231, then using Gasharov's algorithm on can factor $P_w(t) = (1 + t + t^2 + \cdots + t^k)P_v(t)$ for some $v \in S_{n-1}$ avoiding 3412 and 4231. By induction, $P_w(t)$ is palindromic. Use Carrell-Peterson Thm.

Factoring Algorithm: Look for n in one-line notation for w. Because w avoids 3412 and 4231, either n begins a decreasing sequence ending in w_n, or w^{-1} does. Let v be the same as w but with n removed. Partition $\{x \leq w\}$ according to the position of n. Each part of the partition looks like $[id, v]$.

22 Years Later . . .

Consequences of the Lakshmibai-Sandhya Theorem:

1. Testing for smoothness of Schubert varieties can be done in polynomial time, $O(n^4)$.

2. There is an explicit formula for counting the number v_n of smooth Schubert varieties for $w \in S_n$ due to Haiman (see also Bousquet-Mélou+Butler):

$$V(t) = \frac{1 - 5t + 3t^2 + t^2\sqrt{1-4t}}{1 - 6t + 8t^2 - 4t^3}$$
$$= t + 2t^2 + 6t^3 + 22t^4 + 88t^5 + 366t^6 + 1552t^7 + 6652t^8 + O(t^9).$$

3. Many geometrical properties of Schubert varieties are now characterized by pattern avoidance or a variation on this theme.

Next lecture: let tell you about 10 of them!

Future Work

Open Problems.

1. Why are S_4 patterns enough to characterize smoothness.

2. Characterize which smooth Schubert varieties have $[id, w]$ isomorphic to its poset dual. True for 4321 but not true for 45321.

3. Among the self-dual permutations, how can we realize Poincaré duality on the corresponding Schubert variety using Schubert polynomials? (from V. Reiner)