Consequences of the Lakshmibai-Sandhya Theorem; the ubiquity of permutation patterns in Schubert calculus and related geometry

Sara Billey
University of Washington
http://www.math.washington.edu/∼billey

MSJ-SI 2012 Schubert calculus, July 17- 20, 2012

Review of Lecture 1

1. Every Schubert variety $X(w) \subset GL_n/B$ is defined by determinantal equations coming from rank conditions.

2. $X(v) \subset X(w)$ if and only if $v \leq w$ in Bruhat order.

3. (Lakshmibai-Seshadri) The tangent space at v to $X(w)$ has dimension $\#\{t_{ij} : vt_{ij} \leq w\}$.

4. The Bruhat graph on w has vertices indexed by $\{v : v \leq w\}$ and edges between vertices which differ by a transposition.

Lakshmibai-Sandhya Theorem

Fact. There exists a simple criterion for characterizing smooth Schubert varieties using pattern avoidance.

Theorem: Lakshmibai-Sandhya 1990 (see also Haiman, Ryan, Wolper)
X_w is non-singular $\iff$ w has no subsequence with the same relative order as 3412 and 4231.

$$\begin{array}{llccl} & w = 625431 & \text{contains} & 6241 \sim 4231 & \Longrightarrow X_{625431} \text{ is singular} \\ \textit{Example:} & w = 612543 & \text{avoids} & 4231 & \Longrightarrow X_{612543} \text{ is non-singular} \\ & & & \& 3412 & \end{array}$$

22 Years Later . . .

Consequences of the Lakshmibai-Sandhya Theorem.

Many geometrical properties of Schubert varieties are now characterized by pattern avoidance or a variation on this theme.

Let me tell you about 10 of them!

10 Pattern Properties

Property 1. (Carrell-Peterson, Deodhar, Gasharov) (ca 1994)
The following are equivalent

1. X_w is smooth.

2. The Bruhat graph for w is regular and every vertex has degree $\ell(w)$.

3. $\ell(w) = \#\{t_{ij} \leq w\}$.

4. w avoids 3412 and 4231.

5. The Poincare polynomial for w, $P_w(t) = \sum_{v \leq w} t^{l(v)}$ is palindromic.

6. The Poincare polynomial for w factors nicely

$$P_w(t) = \prod_{i=1}^{k} (1 + t + t^2 + \cdots + t^{e_i})$$

Example. $P_{4321}(t) = (1+t)(1+t+t^2)(1+t+t^2+t^3)$

10 Pattern Properties

Prop 1 (Continued).

The following are also equivalent

1. X_w is smooth.

2. w avoids 3412 and 4231.

3. In the inversion hyperplane arrangement defined by $x_i - x_j = 0$ for all $i < j$ such that $w(i) > w(j)$, the generating function

$$R_w(t) = \sum_r q^{d(r)} = \sum_{v \leq w} t^{l(v)} = P_w(t)$$

4. The Kazhdan-Lusztig polynomial $P_{x,w}(t) = 1$ for all $x \leq w$.

5. The Kazhdan-Lusztig polynomial $P_{id,w}(t) = 1$.

See Oh-Postnikov-Yoo (2008), Kazhdan-Lusztig (1980) + Deodhar , Carrell-Peterson,Irving (1988), Braden-MacPherson (2001).

Example. $P_{id,3412}(t) = (1 + t)$

Aside on KL-polys

"Everything you need to know to get started:"

- S_n generated by adjacent transpositions $s_i = t_{i,i+1}$ for $1 \leq i < n$.

- $\mathcal{H}$ = *Hecke algebra* associated to S_n generated by $\{T_1, T_2, \ldots, T_{n-1}\}$ with relations

    1. $(T_i)^2 = (q-1)T_i + q.$
    2. $T_iT_j = T_jT_i \quad$ if $|i-j| > 1.$
    3. $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ for all $1 \leq i < n$.

- $T_w = T_{i_1}T_{i_2}\cdots T_{i_p}$ if the reduced expression $s_{i_1}s_{i_2}\ldots s_{i_p} = w \in S_n$.

Easy Fact. $\{T_w \ : \ w \in W\}$ form a linear basis for $\mathcal{H}$.

An Involution on the Hecke Algebra

Observation. T_w's are invertible over $\mathbb{Z}[q, q^{-1}]$.

- Recall the relation $(T_i)^2 = (q-1)T_i + q$.
- $(T_i)^{-1} = q^{-1}T_i - \left(1 - q^{-1}\right)$.
- $(T_{w^{-1}})^{-1} = (T_{i_1})^{-1} \cdots (T_{i_p})^{-1}$ if $s_{i_1} s_{i_2} \ldots s_{i_p} = w$ (reduced).

Kazhdan-Lusztig Involution. Linear transformation interchanging

$$T_w \overset{i}{\longleftrightarrow} (T_{w^{-1}})^{-1}$$

$$q \overset{i}{\longleftrightarrow} q^{-1}$$

Kazhdan-Lusztig Basis for $\mathcal{H}$

Theorem. (KL, 1979) There exists a unique basis $\{C'_w : w \in W\}$ for the Hecke algebra over $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ such that

1. $i(C'_w) = C'_w$.
2. The change of basis matrix is upper triangular and determined by

$$C'_w = q^{-\frac{1}{2}\ell(w)} \sum_{x \leq w} P_{x,w}(q)\, T_x$$

where $P_{w,w} = 1$ and for all $x < w$, $P_{x,w}(q) \in \mathbb{Z}[q]$ with degree at most

$$\frac{\ell(w) - \ell(x) - 1}{2}.$$

Defn. $P_{x,w}(q)$ is the *Kazhdan-Lusztig polynomial* for x, w.

Examples

$$C'_{s_i} = q^{-\frac{1}{2}}(1 + T_i) = q^{\frac{1}{2}}(1 + T_i^{-1})$$

$$\begin{aligned} C'_{s_i} C'_{s_j} &= q^{-1}(1 + T_i)(1 + T_j) \\ &= q^{-1}(1 + T_i + T_j + T_i T_j) \\ &= C'_{s_i s_j} \qquad \text{for } i \neq j \end{aligned}$$

$$\begin{aligned} C'_{s_1} C'_{s_2} C'_{s_1} &= q^{-\frac{3}{2}}(1 + T_1)(1 + T_2)(1 + T_1) \\ &= q^{-\frac{3}{2}}(1 + 2T_1 + T_2 + T_1 T_2 + T_2 T_1 + T_1^2 + T_1 T_2 T_1) \\ &= q^{-\frac{3}{2}}(1 + 2T_1 + T_2 + T_1 T_2 + T_2 T_1 + ((q-1)T_1 + q) + T_1 T_2 T_1) \end{aligned}$$

$$C'_{s_1 s_2 s_1} = C'_{s_1} C'_{s_2} C'_{s_1} - C'_{s_1}$$

Deodhar Elements

Defn. $w \in S_n$ is *Deodhar* if $C'_w = C'_{s_{i_1}} C'_{s_{i_2}} \cdots C'_{s_{i_p}}$ for some reduced expression $s_{i_1} s_{i_2} \cdots s_{i_p} = w$.

- $C'_{s_1 s_2} = C'_{s_1} C'_{s_2}$ is *Deodhar*.
- $C'_{s_1 s_2 s_1} = C'_{s_1} C'_{s_2} C'_{s_1} - C'_{s_1}$ is *nonDeodhar*.

Kazhdan-Lusztig Polynomials

Observation. We have $P_{x,w} = P_{xs_i,w}$. If $ws < w$ and $xs < x < w$,

$$P_{x,w}(q) = qP_{xs_i,ws_i}(q) + P_{x,ws_i}(q) - \sum_{zs_i<z} q^{\frac{\ell(w)-\ell(z)}{2}} \mu(z, ws_i) P_{x,z}(q).$$

where $\mu(x, w) =$ coefficient of $q^{\frac{\ell(w)-\ell(x)-1}{2}}$ in $P_{x,w}(q)$.

Theorem. (KL,1980)
If W is a Weyl group or affine Weyl group then

$$P_{x,w}(q) = \sum \dim \mathcal{IH}^i_x(X_w)\; q^i.$$

Corollary.
The coefficients of $P_{x,w}(q)$ are non-negative integers with constant term 1.

Examples of Kazhdan–Lusztig polynomials

Below are all $P_{x,w}(q)$ with $x = \mathbf{id}$ and $w \in S_5$ which are different from 1:

w	$P_{\mathrm{id},w}$
(14523) (15342) (24513) (25341) (34125) (34152) (35124) (35142) (35241) (35412) (41523) (42315) (42351) (42513) (42531) (43512) (45132) (45213) (51342) (52314) (52413) (52431) (53142) (53241) (53421) (54231)	$q+1$
(34512) (45123) (45231) (53412)	$2q+1$
(52341)	q^2+2q+1
(45312)	q^2+1

Interesting Properties

1. (Beilinson–Bernstein, Brylinski–Kashiwara, 1981) The multiplicities of irreducibles in the formal character of a Verma module are determined by the $P_{v,w}(1)$.

2. (Irving, 1988, Braden–MacPherson, 2001)

   If $x \leq y$, $\mathbf{coef}_{q^k} P_{x,w}(q) \geq \mathbf{coef}_{q^k} P_{y,w}(q)$

3. (Polo, 1999) Every polynomial with constant term 1 and nonnegative integer coefficients is the KL-poly of some pair of permutations.

4. (McLarnan-Warrington, 2003) Let $\mu(x, w) =$ coefficient of $q^{\frac{\ell(w)-\ell(x)-1}{2}}$. Then for S_9, $\mu(x, w) \in \{0, 1\}$. For S_{10}, $\mu(x, w) = 5$ is possible.

5. (Du Cloux 2003, Brenti 2004, Brenti-Caselli-Marietti 2006) For all Coxeter groups, there exists a formula for $P_{x,w}(q)$ which only depends on the abstract interval $[id, w]$ in Bruhat order.

10 Pattern Properties

Property 1 (Continued). Some localized smoothness tests are similar. The following are equivalent

1. X_w is smooth at $v \in S_n$.
2. $\ell(w) = \{vt_{ij} \leq w\}$. (L-s)
3. The Kazhdan-Lusztig polynomial $P_{x,w}(t) = 1$ for all $v \leq x \leq w$. (K-L)
4. The Kazhdan-Lusztig polynomial $P_{v,w}(t) = 1$ (C-P,I,B-M).

Question. How do 3412 and 4231 patterns help to identify all singular points?

10 Pattern Properties

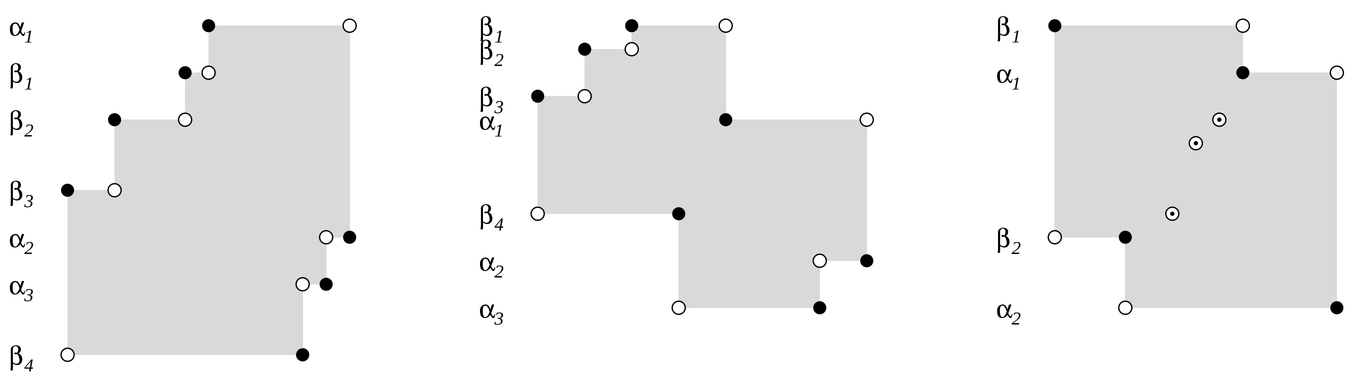
Property 2.

(Billey-Warrington, Manivel,Kassel-Lascoux-Reutenauer,Cortez) (ca 2000)

$\boldsymbol{X_v}$ is an irreducible component of the singular locus of $\boldsymbol{X_w}$ $\iff$

$$\boldsymbol{v} = \boldsymbol{w} \cdot \text{(1-cycle permutation)}$$

corresponding to a 4231 or 3412 or 45312 pattern of the following form



(1) (2) (3)

Here ○'s denote 1's in $\boldsymbol{w}$, ●'s denote 1's in $\boldsymbol{v}$.

10 Pattern Properties

Thm. (Zariski) X is a smooth variety iff the local ring at every point is regular.

Def. X is *factorial* at a point $\iff$ the local ring at that point is a unique factorization domain.

Property 3. (Bousquet-Mélou+Butler, 2007, conj. by Woo-Yong) X_w is factorial at every point $\iff$ w avoids 4231 and $3\underline{41}2$. Here $3\underline{41}2$ means the 4 and 1 must be adjacent.

Thm.(Bousquet-Mélou+Butler) There is an explicit formula for counting the number f_n of factorial Schubert varieties for $w \in S_n$:

$$F(t) = \frac{(1-t)(1-4t-2t^2)-(1-5t)\sqrt{(1-4t)}}{2(1-5t+2t^2-t^3)}$$
$$= x + 2x^2 + 6x^3 + 22x^4 + 89x^5 + 379x^6 + 1661x^7 + 7405x^8 + \dots$$

10 Pattern Properties

Property 4. There exists a simple criterion for characterizing Gorenstein Schubert varieties using modified pattern avoidance.

Def. X is *Gorenstein* if it is Cohen-Macaulay and its canonical sheaf is a line bundle.

Theorem: Woo-Yong (2004) X_w is Gorenstein $\iff$

- w avoids $\mathbf{31542}$ and $\mathbf{24153}$ *with Bruhat restrictions* $\{t_{15}, t_{23}\}$ and $\{t_{15}, t_{34}\}$

- for each descent d in w, the associated partition $\lambda_d(w)$ has all of its inner corners on the same antidiagonal.

Gorenstein Schubert varieties

Sketch of proof.

- Step 1: Schubert varieties are all Cohen-Macaulay. (Ramanathan, 1985)
- Step 2: (Brion, Knutson, Kumar) Testing if X_w is Gor. reduces to a comparison using the Weil divisor class group and the Cartier class group.
- Step 3: The Weil divisor class group is generated by the $[X_v] \in H^*(G/B)$ such that w covers v in Bruhat order $(v < \cdot w)$.
- Step 4: The Cartier class group is generated by $[X_{w_0 s_i}][X_w]$ and

$$[X_{w_0 s_i}][X_w] = \sum [X_{wt_{ab}}]$$

summed over all $wt_{ab} : a \leq i < b,\ \ell(v) = \ell(w) - 1$.

- The Schubert variety X_w is Gorenstein if and only if there exists an integral solution $(\alpha_1, \ldots, \alpha_{n-1})$ to

$$\sum_{i=1}^{n-1} \alpha_i \left(\sum_{v = wt_{ab} : a \leq i < b \ell(v) = \ell(w) - 1} [X_{wt_{ab}}] \right) = \sum_{v < \cdot w} [X_v]$$

10 Pattern Properties

Property 5. (Gasharov-Reiner, 2002)

Def. $X(w)$ is *defined by inclusions* if it can be described as the set of all flags $F_\bullet$ where $F_i \subset E_j$ or $E_i \subset F_j$ form some collection of pairs i, j.

Theorem. (Gasharov-Reiner, 2002) $X(w)$ is *defined by inclusions* iff w avoids 4231, 35142, 42513, 351624.

Theorem. (Hultman-Linusson-Shareshian-Sjöstrand, ca 2007) The number of regions in the inversion arrangement for w is at most the number of elements below w in Bruhat order iff w avoids 4231, 35142, 42513, 351624.

Cohomology of Schubert varieties

Gasharov-Reiner show that Schubert varieties defined by inclusions have a nice presentation for their cohomology ring.

Thm.(Carrell,1992) $H^*(X_w) \approx H^*(G/B)/I_w$ where I_w is generated by all $\mathfrak{S}_v = [X_{w_0 v}]$ such that $v \not\leq w$.

Thm. (Akyldiz-Lacoux-Pragacz, 1992) I_w is generated by $\mathfrak{S}_v$ such that $v \not\leq w$ and v is Grassmannian (at most 1 descent).

Def. x is *bigrassmannian* if both x and x^{-1} are Grassmannian.

Thm.(Reiner-Woo-Yong, 2010) I_w is generated by $\mathfrak{S}_v$ such that $v \not\leq w$, v is Grassmannian and there exists some bigrassmannian $x \in E(w)$ such $x \leq v$ and $\mathbf{Des}(x) = \mathbf{Des}(v)$.

Essential sets

Def. Let $E(w)$ be the set of permutations which are minimal elements in Bruhat order in the complement of the interval $[id, w]$.

Thm. (Lascoux- Schützenberger 1992, Geck-Kim 1997)
The elements in $E(w)$ are bigrassmannian.

Thm.(Reiner-Woo-Yong, 2010)
There exists a bijection between $E(w)$ and Fulton's essential set.

Open. What is the relationship between $E(w)$ and defining equations for Schubert varieties in other types?

Open. Find a minimal set of generators for I_w for all $w \in S_n$. (See Reiner-Woo-Yong conjecture).

10 Pattern Properties

Property 6. (Deodhar, Billey-Warrington, 1998)
The following are equivalent

1. $C'_w = C'_{s_{i_1}} C'_{s_{i_2}} \cdots C'_{s_{i_p}}$ for some/any $s_{i_1} s_{i_2} \cdots s_{i_p} = w$ (reduced).

2. The Bott-Samelson resolution of X_w is small.

3. $\sum_{v \le w} t^{l(v)} P_{v,w}(t) = (1+t)^{l(w)}$.

4. For each $v \le w$, the Kazhdan-Lusztig polynomial

$$P_{v,w}(t) = \sum_{\sigma \in E(v,w)} t^{\mathrm{defect}(\sigma)}.$$

5. w is 321-hexagon avoiding, i.e. avoids

$$321, 56781234, 56718234, 46781235, 46718235$$

10 Pattern Properties

Property 7. Boolean permutations

Theorem.(Tenner, 2006)The principle order ideal below w in Bruhat order is a Boolean lattice $\iff$ w is $\mathbf{321}$ and $\mathbf{3412}$ avoiding.

Equivalently, the Bott-Samelson resolution of $X(w)$ isomorphic to $X(w)$.

Note: Boolean permutations are 321-hexagon avoiding.

Theorem.(Fan '98, West '96). The number of Boolean permutations in S_n is the Fibonacci number F_{2n-1}, e.g. $F_1 = 1, F_3 = 2, F_5 = 5$.

10 Pattern Properties

Property 8. (Woo, 2009)
The Kazhdan-Lusztig polynomial $P_{id,w}(1) = 2 \iff w$ avoids 653421, 632541, 463152, 526413, 546213, and 465132 and the singular locus of X_w has exactly 1 component.

Def. $KL_m = \{w \in S_\infty \mid P_{id,w}(1) \leq m\}$.

Example. KL_1 are the permutations indexing smooth Schubert varieties.

Extension (Billey-Weed): KL_2 is characterized by 66 permutation patterns on 5,6,7 or 8 entries.

Open. KL_m is closed under taking patterns. Can it always be described by a finite set of patterns?

KL_2 Patterns

Extension (Billey-Weed): KL_2 is characterized by 66 permutation patterns on 5,6,7 or 8 entries.

```
;; 44 patterns in S_5,6,7 ;;; 22 more in S_8

  '((4 5 1 2 3) (3 4 5 1 2) (5 3 4 1 2) (5 2 3 4 1) (4 5 2 3 1)
 (3 5 1 6 2 4) (5 2 3 6 1 4) (5 2 6 3 1 4) (6 2 4 1 5 3) (5 2 4 6 1 3)
 (4 6 2 5 1 3) (5 2 6 4 1 3) (5 4 6 2 1 3) (3 6 1 4 5 2) (4 6 1 3 5 2)
 (3 6 4 1 5 2) (4 6 3 1 5 2) (5 3 6 1 4 2) (4 6 5 1 3 2) (4 2 6 3 5 1)
 (6 3 2 5 4 1) (6 3 5 2 4 1) (6 4 2 5 3 1) (6 5 3 4 2 1)
 (3 6 1 2 7 4 5) (6 2 3 1 7 4 5) (6 2 4 1 7 3 5) (3 4 1 6 7 2 5)
 (4 2 3 6 7 1 5) (4 2 6 3 7 1 5) (4 2 6 7 3 1 5) (3 7 1 2 5 6 4)
 (7 2 3 1 5 6 4) (3 7 1 5 2 6 4) (3 7 5 1 2 6 4) (7 5 2 3 1 6 4)
 (6 2 5 1 7 3 4) (7 2 6 1 4 5 3) (3 4 1 7 5 6 2) (3 5 1 7 4 6 2)
 (4 5 1 7 3 6 2) (4 2 3 7 5 6 1) (5 3 4 7 2 6 1) (4 2 7 5 6 3 1)
 (3 4 1 2 7 8 5 6) (4 2 3 1 7 8 5 6) (3 4 1 7 2 8 5 6)
 (4 2 3 7 1 8 5 6) (4 2 7 3 1 8 5 6) (3 5 1 2 7 8 4 6)
 (5 2 3 1 7 8 4 6) (5 2 4 1 7 8 3 6) (3 4 1 2 8 6 7 5)
 (4 2 3 1 8 6 7 5) (3 4 1 8 2 6 7 5) (4 2 3 8 1 6 7 5)
 (4 2 8 3 1 6 7 5) (3 4 1 8 6 2 7 5) (4 2 3 8 6 1 7 5)
 (4 2 8 6 3 1 7 5) (3 5 1 2 8 6 7 4) (5 2 3 1 8 6 7 4)
 (3 6 1 2 8 5 7 4) (6 2 3 1 8 5 7 4) (5 2 4 1 8 6 7 3)
```

10 Pattern Properties

Property 9. (LCI permutations)

Def. A local ring R is a *local complete intersection (lci)* if it is the quotient of some regular local ring by an ideal generated by a regular sequence. A variety is lci if every local ring is lci.

Thm. (Úlfarsson-Woo, 2011) The Schubert variety X(w) is lci if and only if w avoids 53241, 52341, 52431, 35142, 42513, and 426153.

10 Pattern Properties

Property 10. A permutation is *vexillary* if it avoids $\mathbf{2143}$.

Combinatorial Properties.

1. (Edelman-Greene, Macdonald) The number of reduced words for a vexillary permutation v is equal to the number of standard tableaux of shape determined by sorting the lengths of the rows of the diagram of v.

2. The Stanley symmetric function F_v is a Schur function iff v is vexillary.

$$F_v = \sum_{\mathrm{a}=a_1a_2\ldots a_k \in R(v)} \sum_{i_1 \le \cdots \le i_k \in C(\mathrm{a})} x_{i_1} x_{i_2} \cdots x_{i_k}$$

   where $R(v)$ are the reduced words for v and $C(\mathrm{a})$ are the weakly increasing sequences of positive integers such that $i_j < i_{j+1}$ if $a_j < a_{j+1}$.

3. (Tenner, 2006) The permutation v is vexillary iff for every permutation w containing v, there exists a reduced decomposition $\mathrm{a} \in R(w)$ containing a shift of some $\mathrm{b} \in R(v)$ as a factor.

10 Pattern Properties

Property 10. A permutation is *vexillary* if it avoids $\mathbf{2143}$. (Lascoux-Schützenberger,1984)

Geometric Properties.

1. (Fulton, 1992) The *essential set* for w, the cells in the diagram of w with no neighbor directly east or north, lie on an decreasing piecewise linear curve.

2. (Lascoux, 1995) There exists a combinatorial approach to computing the Kazhdan-Lusztig polynomials $P_{v,w}$ when w is *covexillary*, i.e. 3412-avoiding.

3. (Li-Yong, ca 2011) There exists a combinatorial rule for computing multiplicities for $X(w)$ when w is 3412-avoiding.

10 Pattern Properties

Property 10. (continued) k-vexillary permutations.

Def. A permutation w is *k-vexillary* if its Stanley symmetric function F_w has at most k terms in its expansion.
Example: $F_{2143} = s_{(2)} + s_{(1,1)}$, so 2143 is 2-vexillary.

Thm. (Billey-Pawlowski) Let w be a permutation.

1. w is 2-vexillary iff w avoids 35 patterns in S_5, S_6, S_7, S_8.

2. w is 3-vexillary iff w avoids 91 patterns in S_5, S_6, S_7, S_8.

Conjecture. The k-vexillary permutations form an order ideal in the poset on all permutations ordered by pattern containment.

10 Pattern Properties

List of 2-vexillary patterns:

```
(setf *2-vex*
'((3 2 1 5 4) (2 1 5 4 3)

  (2 1 4 3 6 5) (2 4 1 3 6 5) (3 1 4 2 6 5)  (3 1 2 6 4 5)
  (2 1 4 6 3 5) (2 4 1 6 3 5) (2 3 1 5 6 4) (2 1 5 3 6 4)

  (3 1 5 2 6 4) (4 2 6 1 5 3) (5 2 7 1 4 3 6) (5 1 7 3 2 6 4)
  (4 2 6 5 1 7 3) (2 5 4 7 1 6 3) (5 4 7 2 1 6 3) (5 2 7 6 1 4 3)

  (6 1 8 3 2 5 4 7) (2 6 4 8 1 5 3 7) (6 4 8 2 1 5 3 7) (2 6 5 8 1 4 3 7)
  (6 5 8 2 1 4 3 7) (5 1 7 3 6 2 8 4) (5 1 7 6 3 2 8 4) (6 1 8 3 7 2 5 4)
  (6 1 8 7 3 2 5 4) (2 5 4 7 6 1 8 3) (5 4 7 2 6 1 8 3) (5 4 7 6 2 1 8 3)
  (2 6 4 8 7 1 5 3) (6 4 8 7 2 1 5 3) (2 6 5 8 7 1 4 3) (6 5 8 2 7 1 4 3)
  (6 5 8 7 2 1 4 3)))
```

10 Pattern Properties

List of 3-vexillary patterns:

```
(setf *3-vex*
'((2 1 4 3 6 5) (4 3 2 1 6 5) (3 2 1 6 4 5) (4 2 1 6 3 5) (3 2 1 5 6 4)
  (3 2 5 1 6 4) (3 1 2 6 5 4) (2 3 1 6 5 4) (3 2 1 6 5 4) (3 1 6 2 5 4)
  (3 2 6 1 5 4) (2 4 1 6 5 3) (4 2 1 6 5 3) (4 2 6 1 5 3) (2 1 6 5 4 3)
(3 5 2 1 4 7 6) (4 2 5 1 3 7 6) (2 5 4 1 3 7 6) (5 2 4 1 3 7 6)
(4 3 1 5 2 7 6) (3 5 1 4 2 7 6) (5 3 1 4 2 7 6) (4 1 5 3 2 7 6)
(3 5 2 4 1 7 6) (4 2 5 3 1 7 6) (2 4 1 3 7 5 6) (3 1 4 2 7 5 6)
(3 1 2 5 7 4 6) (2 3 1 5 7 4 6) (2 5 1 3 7 4 6) (3 1 5 2 7 4 6)
(3 5 1 2 7 4 6) (2 4 1 5 7 3 6) (2 5 1 4 7 3 6) (2 5 4 1 7 3 6)
(2 1 5 7 4 3 6) (2 5 1 7 4 3 6) (5 2 7 1 4 3 6) (2 4 1 3 6 7 5)
(3 1 4 2 6 7 5) (3 1 2 6 4 7 5) (2 3 1 6 4 7 5) (3 1 6 2 4 7 5)
(2 4 1 6 3 7 5) (3 1 4 6 2 7 5) (3 4 1 6 2 7 5) (3 1 6 4 2 7 5)
(4 1 6 3 2 7 5) (3 1 6 2 7 4 5) (2 4 1 6 7 3 5) (2 1 6 4 7 3 5)
(2 1 4 7 6 3 5) (2 1 7 4 6 3 5) (2 1 6 5 3 7 4) (3 1 6 5 2 7 4)
(2 1 5 7 3 6 4) (2 1 7 5 3 6 4) (5 1 7 3 2 6 4) (2 1 6 3 7 5 4)
(4 2 6 5 1 7 3) (2 1 5 7 4 6 3) (2 5 4 7 1 6 3) (5 4 7 2 1 6 3)
(2 1 6 4 7 5 3) (5 2 7 6 1 4 3)

(2 4 6 1 3 5 8 7) (4 1 5 2 6 3 8 7) (4 1 2 3 8 5 6 7) (2 1 4 6 8 3 5 7)
(2 4 6 1 8 3 5 7) (6 1 8 3 2 5 4 7) (2 6 4 8 1 5 3 7) (6 4 8 2 1 5 3 7)
(2 6 5 8 1 4 3 7) (6 5 8 2 1 4 3 7) (3 4 1 2 7 8 5 6)
(2 3 4 1 6 7 8 5) (2 1 6 3 7 4 8 5) (4 1 6 2 7 3 8 5)
(5 1 7 3 6 2 8 4) (5 1 7 6 3 2 8 4) (6 1 8 3 7 2 5 4)
(6 1 8 7 3 2 5 4) (2 5 4 7 6 1 8 3) (5 4 7 2 6 1 8 3)
(5 4 7 6 2 1 8 3) (2 6 4 8 7 1 5 3) (6 4 8 7 2 1 5 3)
(2 6 5 8 7 1 4 3) (6 5 8 2 7 1 4 3) (6 5 8 7 2 1 4 3))
)
```

Future Work

Open Problems.

1. Characterize the Gorenstein, lci and factorial locus of $X(w)$ using patterns. (Woo-Yong)

2. (From Úlfarsson) Is there a nice generating function to count the number of Gorenstein/lci permutations or Schubs defined by inclusions, etc.

3. Find a geometric explanation why a finite number of patterns suffice in all cases above.

4. What nice properties does the inversion arrangement have for other pattern avoiding families?

5. KL_m is closed under taking patterns. Can it always be described by a finite set of patterns?

6. Conjecture (Woo): The Schubert varieties with multiplicity ≤ 2 can be characterized by pattern avoidance.

7. What other filtrations on the set of all permutations can be characterized by (generalized) patterns?