Consequences of the Lakshmibai-Sandhya Theorem; the ubiquity of permutation patterns in Schubert calculus and related geometry

Sara Billey
University of Washington
http://www.math.washington.edu/∼billey/japan

MSJ-SI 2012 Schubert calculus, July 17- 20, 2012

Review of Lecture 1: Classical Results

1. Every Schubert variety $X(w) \subset GL_n/B$ is defined by determinantal equations coming from rank conditions.

2. $X(v) \subset X(w)$ if and only if $v \leq w$ in Bruhat order.

3. (Lakshmibai-Seshadri) The tangent space at v to $X(w)$ has dimension $\#\{t_{ij} : vt_{ij} \leq w\}$.

4. The Bruhat graph on w has vertices indexed by $\{v : v \leq w\}$ and edges between vertices which differ by a transposition.

5. (Lakshmibai-Sandhya) X_w is smooth iff w avoids 3412 and 4231.

Review of Lecture 2: Properties Defined By Patterns

1. Smooth permutations: 3412 and 4231 avoiding.
2. Permutation patterns determine the irreducible components of singular loci of Schubert varieties.
3. Factorial Schubert varieties: $\mathbf{4231}$ and $\mathbf{3\underline{41}2}$ avoiding.
4. Gorenstein Schubert varieties: $\mathbf{31542}$ and $\mathbf{24153}$ *with Bruhat restrictions* plus Grassmannian condition.
5. Schubert varieties "defined by inclusions": 4231, 35142, 42513, 351624.
6. Deodhar permutations/ 321-hexagon avoiding: 321, 56781234, 56718234, 46781235, 46718235.
7. Boolean permutations: $\mathbf{321}$ and $\mathbf{3412}$ avoiding.
8. KL_2 permutations: 653421, 632541, 463152, 526413, 546213, and 465132 and the singular locus of X_w has exactly 1 component.
9. LCI permutations: 53241, 52341, 52431, 35142, 42513, and 426153.
10. Vexillary permutations: 2143 avoiding.

Pattern Avoidance For Any Coxeter Group

Outline.

1. Coxeter Groups
2. Generalized Pattern Avoidance
3. Applications
4. Open Problems

Notation

- G = *Coxeter graph* with vertices $\{1, 2, \ldots, n\}$, edges labeled by $\mathbb{Z}_{\geq 3} \cup \infty$.

$$\bullet_1 \overset{4}{\text{——}} \bullet_2 \overset{3}{\text{——}} \bullet_3 \overset{3}{\text{——}} \bullet_4 \quad \approx \quad \bullet_1 \overset{4}{\text{——}} \bullet_2 \text{——} \bullet_3 \text{——} \bullet_4$$

- W = *Coxeter group* generated by $\{s_1, s_2, \ldots, s_n\}$ with relations
  1. $s_i^2 = 1.$
  2. $s_i s_j = s_j s_i$ if i, j not adjacent in G.
  3. $\underbrace{s_i s_j s_i \cdots}_{m(i,j) \text{ gens}} = \underbrace{s_j s_i s_j \cdots}_{m(i,j) \text{ gens}}$ if i, j connected by edge labeled $m(i, j)$.

Examples

Dihedral groups: $\mathbf{Dih_{10}}$ $\qquad \bullet_{\mathbf{1}} \overset{\mathbf{5}}{\text{——}} \bullet_{\mathbf{2}}$

Symmetric groups: $\boldsymbol{S_5}$ $\qquad \bullet_{\mathbf{1}} \text{ —— } \bullet_{\mathbf{2}} \text{ —— } \bullet_{\mathbf{3}} \text{ —— } \bullet_{\mathbf{4}}$

Hyperoctahedral groups: $\boldsymbol{B_4}$ $\qquad \bullet_{\mathbf{1}} \overset{4}{\text{——}} \bullet_{\mathbf{2}} \text{ —— } \bullet_{\mathbf{3}} \text{ —— } \bullet_{\mathbf{4}}$

$\boldsymbol{E_8}$: $\qquad \bullet_{\mathbf{1}} \text{ —— } \bullet_{\mathbf{2}} \text{ —— } \bullet_{\mathbf{3}} \text{ —— } \bullet_{\mathbf{4}} \text{ —— } \bullet_{\mathbf{5}} \text{ —— } \bullet_{\mathbf{6}} \text{ —— } \bullet_{\mathbf{7}}$

$\bullet_{\mathbf{8}}$

Notation

- W = *Coxeter group* generated by $S = \{s_1, s_2, \ldots, s_n\}$ with special relations.
- R = *Reflections*$= \bigcup_{w \in W} wSw^{-1}$.
- $\ell(w)$ = *length* of w = length of a reduced expression for w.
- *Bruhat order*: $x \leq y \iff \ell(x) < \ell(y)$ and $xy^{-1} \in R$.
- Observation(Chevalley): $x \leq y$ if $y = s_{i_1} s_{i_2} \ldots s_{i_p}$ (*reduced expression*) and $x = s_{i_1}^{\sigma_1} s_{i_2}^{\sigma_2} \ldots s_{i_p}^{\sigma_p}$ for some *mask* $\sigma_1 \ldots \sigma_p \in \{0,1\}^p$.

Mozes Numbers Game

Algorithm. Generates canonical representative for each element in a Coxeter group using its graph.
(See Mozes 1990, Eriksson-Eriksson 1998, Björner-Brenti Book)

Input: Coxeter graph G and expression $s_{i_1} s_{i_2} \ldots s_{i_p} = w$.

Start: Each vertex of graph G assigned value 1. Replace each edge (i, j) of G by two opposing directed edges labeled $f_{ij} > 0$ and $f_{ji} > 0$ so that $f_{ij} f_{ji} = 4\cos^2\left(\frac{\pi}{m(i,j)}\right)$ or $f_{ij} f_{ji} = 4$ if $m(i, j) = \infty$.

Good choices:

$m(i,j)$	f_{ij}	f_{ji}
3	1	1
4	2	1
6	3	1

Mozes Numbers Game

Loop. For each s_{i_k} in $s_{i_1} s_{i_2} \ldots s_{i_p}$ *fire node* i_k.

To fire node i, add to the value of each neighbor j the current value at node i multiplied by f_{ij}. Negate the value on node i.

Output.: $G(w) =$ the final values on the nodes of G.

Mozes Numbers Game

Loop. For each s_{i_k} in $s_{i_1} s_{i_2} \ldots s_{i_p}$ *fire node* i_k.

To fire node i, add to the value of each neighbor j the current value at node i multiplied by f_{ij}. Negate the value on node i.

Output.: $G(w) =$ the final values on the nodes of G.

Properties:

1. Output only depends on the product $s_{i_1} s_{i_2} \ldots s_{i_p}$ and not on the particular choice of expression.

2. Node i is negative in $G(w)$ iff $ws_i < w$.

3. Node i never has value 0.

4. If $I \subset S$, modify the game to get representatives for W/W_I by starting with initial value 0 on nodes in I. Then $ws_i = w$ iff node i has value 0. Useful for Grassmannians and affine Grassmannians.

Linear Reps for Coxeter groups

Associate to a Coxeter group $\boldsymbol{W}$ a "*root system*" $\boldsymbol{\Phi} \subset \boldsymbol{V} = \mathbb{R}^{|\boldsymbol{S}|}$ such that

1. $\{\boldsymbol{\alpha_s} : \boldsymbol{s} \in \boldsymbol{S}\}$ forms an orthogonal basis of $\boldsymbol{V}$.
2. $\boldsymbol{W}$ acts linearly on $\boldsymbol{V}$, and $\boldsymbol{\Phi}$ is $\boldsymbol{W}$-invariant.
3. $\boldsymbol{\Phi}_+ =$ *positive roots* $= \{\boldsymbol{\alpha} \in \boldsymbol{\Phi} : \boldsymbol{\alpha} = \sum \boldsymbol{c_s \alpha_s},\ \boldsymbol{c_s} \geq \boldsymbol{0}\}$,
   $\boldsymbol{\Phi}_- =$ *negative roots* $= \{\boldsymbol{\alpha} \in \boldsymbol{\Phi} : \boldsymbol{\alpha} = \sum \boldsymbol{c_s \alpha_s},\ \boldsymbol{c_s} \leq \boldsymbol{0}\}$, then
   $\boldsymbol{\Phi} = \boldsymbol{\Phi}_+ \cup \boldsymbol{\Phi}_-$ (disjoint).

Reflection Representations for Coxeter groups

Associate to a Coxeter group W a "*root system*" $\Phi \subset V = \mathbb{R}^{|S|}$ such that

1. $\{\alpha_s : s \in S\}$ forms a basis of V.

2. W acts linearly on V, and Φ is W-invariant.

3. $\Phi_+ =$ *positive roots* $= \{\alpha \in \Phi : \alpha = \sum c_s \alpha_s,\ c_s \geq 0\}$,
   $\Phi_- =$ *negative roots* $= \{\alpha \in \Phi : \alpha = \sum c_s \alpha_s,\ c_s \leq 0\}$, then
   $\Phi = \Phi_+ \cup \Phi_-$ (disjoint).

4. Bijection: $\alpha : R \longleftrightarrow \Phi_+$.

5. For $r \in R, w \in W$, $rw > w \iff \alpha_r \in w\Phi_+$.

(See construction in Björner-Brenti: Combinatorics of Coxeter groups.)

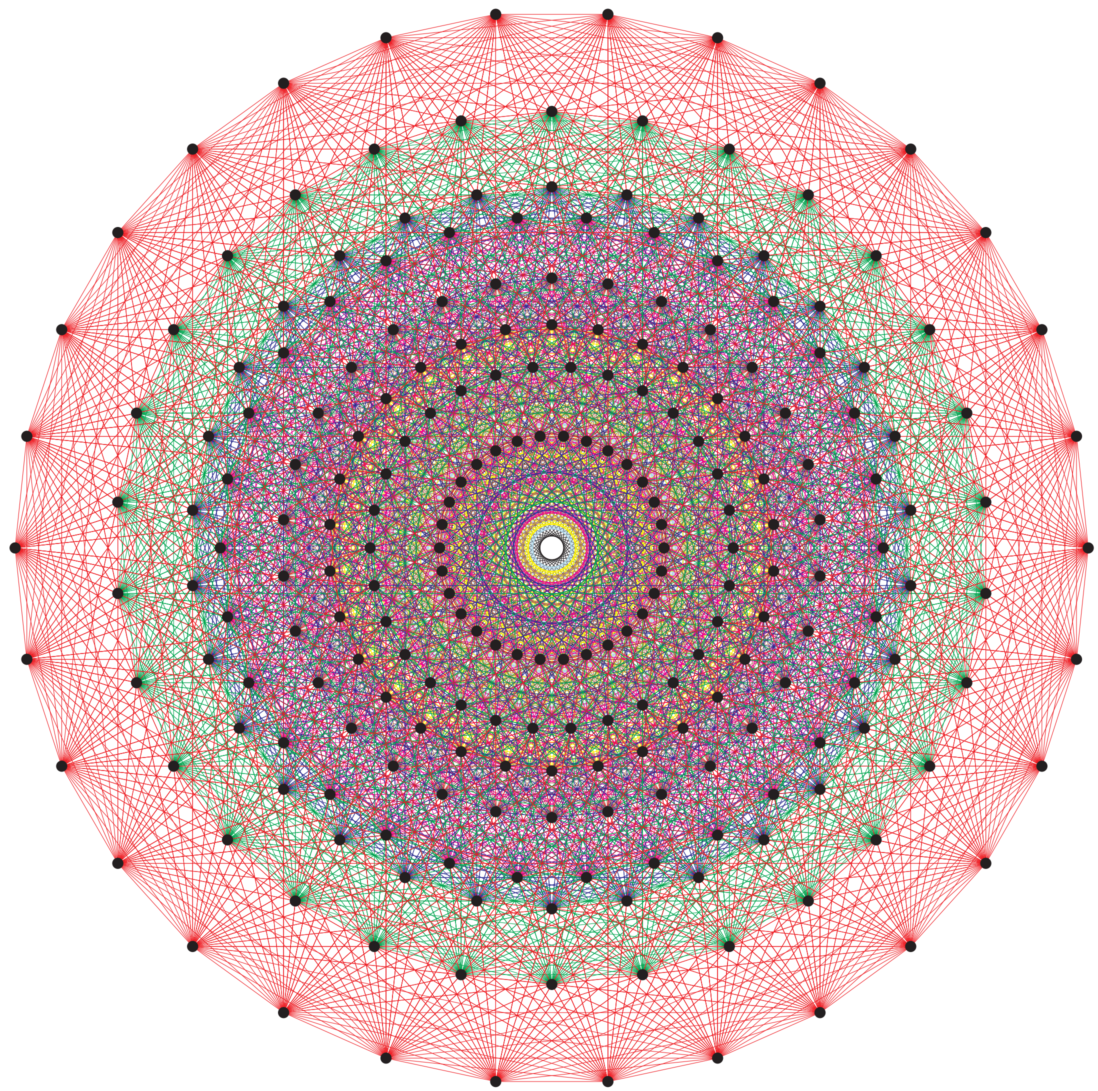
Examples

Assume $e_1, \ldots, e_n$ is the standard orthonormal basis of $\mathbb{R}^n$.

- A_{n-1}: $\Phi_+ = \{e_i - e_j : i < j\}$
- B_n: $\Phi_+ = \{e_i - e_j : i < j\} \cup \{e_i + e_j : i < j\} \cup \{e_i : i\}$
- C_n: $\Phi_+ = \{e_i - e_j : i < j\} \cup \{e_i + e_j : i < j\} \cup \{2e_i : i\}$
- D_n: $\Phi_+ = \{e_i - e_j : i < j\} \cup \{e_i + e_j : i < j\}$

Examples

John Stembridge's rendering of the root system for $\boldsymbol{E_8}$ projected from $\mathbb{R}^{\mathbf{8}}$. Edges connect nearest neighbors. Color determined by furthest distance of a pair to 0.



Inversion Sets

- For $r \in R, w \in W$, $rw > w \iff \alpha_r \in w\Phi_+$.
- The analog of the *inversion set* is $w\Phi_+ \cap \Phi_-$.

Def. If $H : V \longrightarrow \mathbb{R}$ is a linear function,

$$\Pi_H = \{\alpha \in \Phi : H(\alpha) > 0\}.$$

H is *generic* if $H(\alpha) \neq 0 \ \forall \alpha \in \Phi$.

Example. If $H_1 : V \longrightarrow \mathbb{R}$ defined by $H_1(\alpha_s) = 1 \forall s \in S$, then $\Pi_{H_1} = \Phi_+$.

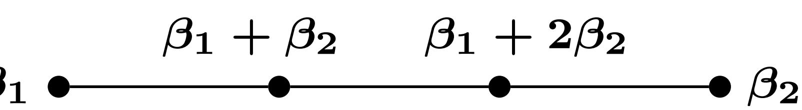
Def. Set $H_w = H_1 \circ w^{-1}$ for all $w \in W$. Then, $\Phi_{H_w} = w\Phi_+$.

Inversion Sets

Key Fact. If $\boldsymbol{H}$ is generic, then $\boldsymbol{\Pi_H = w\Phi_+}$ for some unique $\boldsymbol{w \in W}$.

Below are the positive roots for two types of Coxeter groups drawn projectively in 2-d. Intersecting each picture with a half spaces, identifies an inversion set.

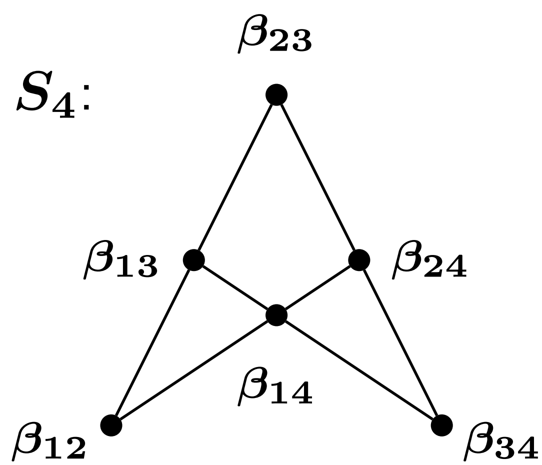
$\boldsymbol{B_2}$:



$\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_1 + 2\beta_2}$ are long roots

$\boldsymbol{\beta_1 + \beta_2}$ and $\boldsymbol{\beta_2}$ are short roots

$\boldsymbol{A_3 = S_4}$:



$$\boldsymbol{\beta_{12} = \beta_1,\ \ \beta_{23} = \beta_2,\ \ \beta_{34} = \beta_3,}$$

$$\boldsymbol{\beta_{13} = \beta_1 + \beta_2,\ \ \beta_{24} = \beta_2 + \beta_3,}$$

$$\boldsymbol{\beta_{14} = \beta_1 + \beta_2 + \beta_3}$$

$D-4$ embedding

Root subsystems

Def. If $U \subset V$ is a subspace, then

- $\Phi^U = \Phi \cap U$ is a *root subsystem* of Φ.
- W^U = group generated by reflections r_α for $\alpha \in \Phi^U$.
- $R^U = R \cap W^U$

Fact. W^U is a Coxeter group with simple reflections $S^U = xIx^{-1}$ for some $I \subset S$ and $x \in W$. Any subgroup of this form is a *parabolic subgroup*.

Coxeter Group Patterns

Def. Given a subspace $U \in V$, we get a *pattern map* or a *flattening map*

$$\mathrm{fl}_U : W \longrightarrow W^U$$

given by mapping w to the unique element $x \in W^U$ such that

$$\begin{aligned} w\Phi_+ \cap U &= \{\alpha \in U \cap \Phi : H_w(\alpha) > 0\} \\ &= \{\alpha \in \Phi^U : H'(\alpha) > 0\} \text{ where } H' = H_w|_U \\ &= x\Phi_+^U. \end{aligned}$$

Coxeter Group Patterns

Def. Given a subspace $U \in V$, we get a *pattern map* or a *flattening map*

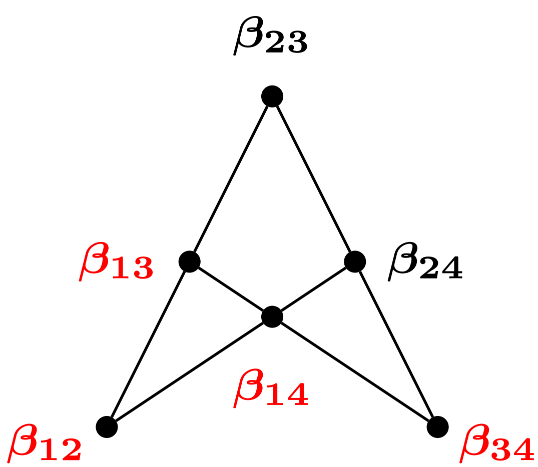
$$\mathrm{fl}_U : W \longrightarrow W^U$$

given by mapping w to the unique element $x \in W^U$ such that

$$\begin{aligned} w\Phi_+ \cap U &= \{\alpha \in U \cap \Phi : H_w(\alpha) > 0\} \\ &= \{\alpha \in \Phi^U : H'(\alpha) > 0\} \text{ where } H' = H_w|_U \\ &= x\Phi_+^U. \end{aligned}$$

Example.

$w = 2431$:



$U = \mathrm{span}\langle \beta_{34}, \beta_{23} \rangle$

Coxeter Patterns

Def. Given a subspace $U \in V$, we get a *pattern map* or a *flattening map*

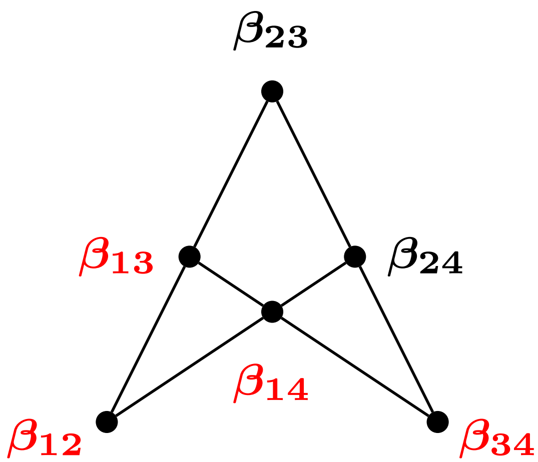
$$\mathrm{fl}_U : W \longrightarrow W^U$$

given by mapping w to the unique element $x \in W^U$ such that

$$\begin{aligned} w\Phi_+ \cap U &= \{\alpha \in U \cap \Phi : H_w(\alpha) > 0\} \\ &= \{\alpha \in \Phi^U : H'(\alpha) > 0\} \text{ where } H' = H_w|_U \\ &= x\Phi_+^U. \end{aligned}$$

Example.

$w = 2431$:



$U = \mathrm{span}\langle \beta_{34}, \beta_{23} \rangle$

$\mathrm{fl}(w) = \mathrm{fl}(243) = 132$

Coxeter Patterns

Thm. (Billey-Braden)

1. $\mathbf{fl}_U$ is W^U-equivariant: $\mathbf{fl}(wx) = w\ \mathbf{fl}(x)\ \forall\ w \in W^U,\ x \in W.$

2. If $\mathbf{fl}(x) \leq^U \mathbf{fl}(wx)$ in Bruhat order on W^U for some $w \in W^U$, then $x \leq wx$ in W.

3. $\mathbf{fl}_U$ is the unique map with properties (1) and (2).

Applications of Coxeter Patterns

Notation.

- G= Semisimple Lie group over $\mathbb{C}$
- B = Borel subgroup
- $T \subset B$ maximal torus.
- $W = N(T)/T$ = Weyl group for G (a finite Coxeter group)

The finite Weyl groups/root systems that arise this way have been completely classified into types $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$.

Bruhat Decomposition. $G = \bigcup_{w \in W} BwB$.

Applications of Coxeter Patterns

Notation.

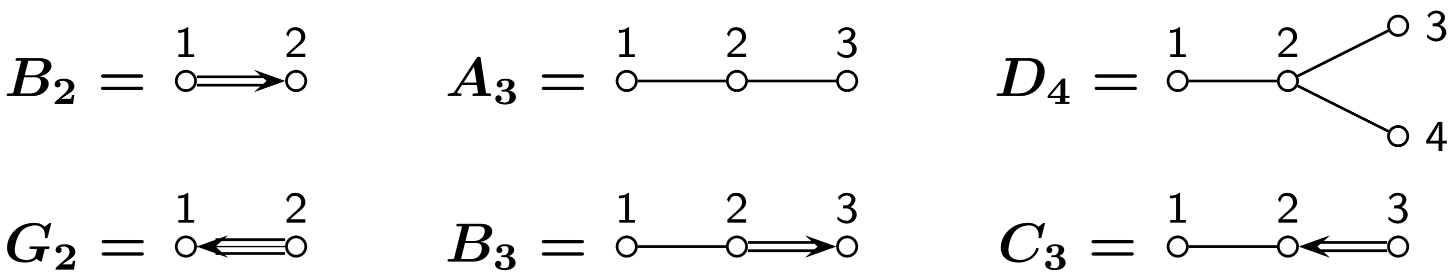
- G/B = (generalized) flag manifold
- $C_w = B \cdot w$ = Schubert cell
- $X_w = \overline{B \cdot w}$ = Schubert variety

Thm. (Billey-Postnikov, 2006)
X_w is smooth $\iff$ for every *stellar* parabolic subgroup W^U, the Schubert variety indexed by $\mathbf{fl}_U(w)$ is smooth in the corresponding flag manifold corresponding with U.

Applications of Coxeter Patterns

Thm. (Billey-Postnikov, 2006)
X_w is smooth $\iff$ for every *stellar* parabolic subgroup W^U,
. $X(\mathrm{fl}_U(w))$ is smooth in G^U/B^U.

Def. W^U is *stellar* if its Coxeter graph has one central vertex v and all other vertices are only adjacent to v.



Dynkin diagrams of stellar root systems

2 patterns in A_3, 1 pattern in B_2, 6 patterns of type B_3 and C_3, 1 pattern of type D_4, 5 patterns of type G_2.

Minimal Patterns

Example. In type B_n using just classical pattern avoidance on signed permutations, the smooth Schubert varieties are classified by avoiding

```
(-2 -1)
(1 2 -3) (1 -2 -3) (-1 2 -3) (2 -1 -3) (-2 1 -3) (3 -2 1)
(2 -4 3 1) (-2 -4 3 1) (3 4 1 2) (3 4 -1 2) (-3 4 1 2)
(4 1 3 -2) (4 -1 3 -2) (4 2 3 1) (4 2 3 -1) (-4 2 3 1)))
```

All length 4 patterns come from A_3 root subsystems.

Example. In type D_4, there are 49 singular Schubert varieties, only does not comes from A_3 root subsystems: $w = s_2 \cdot s_1 s_3 s_4 \cdot s_2 = \bar{1}4\bar{3}2$.

$\mathbf{Sing}\ X(s_2 s_1 s_3 s_4 s_2) = X(s_2)$
$\mathbf{Sing}\ X(s_2 s_1 s_3 s_2) = X(s_2)$ (3412 case)
$\mathbf{Sing}\ X(s_3 s_1 s_2 s_1 s_3) = X(s_1 s_3)$ (4231 case)

Rational Smoothness

Observation. The definition of a Kazhdan-Lusztig polynomial $P_{v,w}(t)$ easily generalizes to all Coxeter groups.

Def. A point $v \in X_w$ is rationally smooth iff $P_{v,w}(t) = 1$.

Smooth $\implies$ rationally smooth.

Thm. (Deodhar, Peterson) For types A, D, E, X_w is smooth iff it's rationally smooth.

Note, by (Mitchell,Billey-Crites) not true for $\widetilde{(A)}_n$. So, need a stronger condition than just *simply laced* (all edges in Coxeter graph have label 3).

Rational Smoothness

Thm. (Carrell-Peterson, Jantzen) The following are equivalent

1. X_w is rationally smooth at v.
2. $P_{v,w}(t) = 1$
3. Bruhat graph on $[v, w]$ is regular of degree $l(w) - l(v)$.

Thm. The following are equivalent

1. X_w is rationally smooth.
2. $P_{id,w}(t) = 1$
3. $P_w(t) = \sum_{v \leq w} t^{\ell(v)}$ is palindromic. (C-P)
4. $P_w(t) = \prod(1 + t + t^2 + \cdots + t^{e_i})$
   (conj McGovern, Akyildiz-Carrell 2010)

Rational Smoothness

Thm. (Billey-Postnikov, 2006) X_w is rationally smooth $\iff$ for every *stellar* parabolic subgroup W^U, $X(\mathrm{fl}_U(w))$ is rationally smooth in G^U/B^U.

Minimal patterns: 2 patterns in A_3, 6 patterns of type B_3 and C_3, 1 pattern of type D_4.

Remark. The rationally smooth but not smooth patterns are only the 1 pattern in B_2 and 5 patterns of type G_2.

Outline of proof

- Step 1: For classical types B, C, D, use Lakshmibai's characterization of the tangent space basis.
- Step 2: Use an analog of Gasharov's theorem to the factor Poincaré polynomial for any signed permutation not containing a singular pattern.
- Step 3: Use Kumar's criterion for (rational) smoothness in the nil-Hecke ring to test G_2 and F_4 by computer.
- Step 4: Run a massive parallel computer on the 696,729,600 elements $w \in E_8$.
  - If w has a pattern from type A or D, calculate the number the coefficient of t^1 and $t^{\ell(w)-1}$ and compare, if different, w is done. If not, calculate the coefficient of t^2 and $t^{\ell(w)-2}$, etc. Eventually one pair differed in every case.
  - If w avoids all patterns from type A or D, use analog of Gasharov's algorithm for factoring $P_w(t)$.

Outline of proof

Question. What is the value of a computer proof?

Outline of proof

Question. What is the value of a computer proof?

Answer. We know the statement is true!

Outline of proof

Question. What is the value of a computer proof?

Answer. We know the statement is true!

Furthermore, we find miracles happen which make computation possible! See for example, the proof of the 4 Color Theorem of Robertson, Sanders, Seymour, Thomas (1997).

Things we learned:

- Only 99.989% of cases only required checking first coefficient.
- To find a factored form for $P_w(t)$ only need to consider quotients using leaf nodes in the Coxeter graph.
- Conjecture: one only needs to check at most n coefficients in to detect the non-palindromic property for any rank n Weyl group. (See Richmond-Slofstra 2012)

Kazhdan-Lusztig Values

Notation.

- W = Weyl group or affine Weyl group
- W^U = parabolic subgroup of W associated to a vector space U
- $M(x, w; U)$ = maximal elements in $[id, w] \cap W^U x$ with respect to a new partial order $\leq_x$

$$wx \leq_x w'x \iff \text{fl}(wx) \leq^U \text{fl}(w'x).$$

Thm. (Billey-Braden) If $x, w \in W$, then

$$P_{x,w}(1) \geq \sum_{y \in M(x,w;U)} P_{y,w}(1) P^U_{\text{fl}(x),\text{fl}(y)}(1).$$

Cor. $P_{x,w}(1) \geq P^U_{\text{fl}(x),\text{fl}(y)}(1)$.

Pattern geometry

Thm.(Billey-Braden) If $X^U_{\mathrm{fl}(w)}$ is singular, then X_w is singular.

Proof Outline.

Realize G^U/B^U as the fixed points of a certain torus action.

Use a theorem of Fogarty-Norman saying that for all smooth algebraic T-schemes X the fixed point scheme X^G is smooth.

Cor. If w contains a pattern $v \in W^U$ and X^U_v is not smooth, then X_w cannot be smooth.

Pattern avoidance in Coxeter groups

1. (Stembridge ca 1998) Characterized the fully commutative elements in types B,D with signed patterns.

2. (R.Green, 2002) 321-avoiding elements in affine Weyl groups.

3. (Reading, 2005) Characterized Coxeter-sortable elements and showed they are equinumerous with clusters and with noncrossing partitions.

4. (Billey-Jones, 2008) Deodhar elements for all Weyl groups.

5. (Billey-Crites, 2012) The rationally smooth Schubert varieties in the affine type A flag manifold are characterized as 3412, 4231 avoiding plus one extra family of twisted spiral varieties.

6. (Chen-Crites-Kuttler, preprint) An affine Schubert variety X_w is smooth $\iff$ $w \in \widetilde{S}_n$ avoids $\mathbf{3412}$ and $\mathbf{4231}$. Furthermore, the tangent space to X_w at the identity can be described in terms of reflection over real and imaginary roots.

7. (Matthew Samuel, preprint) An affine Schubert varieties for all types can be characterized by patterns using a new version of pattern avoidance for Coxeter groups based on reflection groups.

Open Problems

1. Describe the maximal singular locus of a Schubert variety for other semisimple Lie groups using generalized pattern avoidance.

2. Give a pattern based algorithm to produce the factorial and/or Gorenstein locus of a Schubert variety in other types.

3. Is there a nice generating function to count the number of smooth, factorial and/or Gorenstein permutations in other types?

4. Find a geometric explanation why a finite number of patterns suffice in all cases above.

5. What is the right notion of patterns for GKM spaces?

6. Say X_w is *combinatorially smooth* if $\ell(w) = \#\{t_{ij} : t_{ij} \leq w$. Conjecture: the combinatorially smooth elements characterized by generalized pattern avoidance.