Consequences of the Lakshmibai-Sandhya Theorem; the ubiquity of permutation patterns in Schubert calculus and related geometry

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Review

Summary.

1. Classical results on Schubert varieties in flag manifolds
2. 10 Properties characterized by pattern avoidance
3. Generalized pattern avoidance for all Coxeter groups

Computer Tools for Schubert Geometry

Outline:

1. OEIS = Sloane's Online Encyclopedia of Integer Sequences
2. Tenner's database: http://math.depaul.edu/bridget/patterns.html
3. Learning algorithm for patterns
4. Brief intro to SAGE
5. Úlfarsson's recent learning algorithms on marked mesh patterns.

OEIS

In the 1960's, Neil Sloane started collecting integer sequences on punch cards.

1. Today it contains at least 212,892 sequences.
2. Universal database on sequences.
3. Integer sequences are independent of language, notation, and human variation. Makes it easy to find references and connections to other research.
4. Everyone can contribute!
5. Search by keywords or by sequence: 1,2,6,22,89,379.

Demonstration

Try searching OEIS for

1. The number of Gorenstein permutations in S_{10}.
2. The generating function for the number of factorial permutations.
3. The recurrence relation for 321-hexagon avoiding permutations.
4. Is there a generating function for the number of Schubert varieties in the flag manifold of rank n which are locally complete intersections?

Example

Question. Does the following formula have a closed form?

$$\begin{aligned} &\sum_{i=2}^{n+1} \binom{n+3}{i} (i-1)(n+2-i) \\ &+ \sum_{i=1}^{n} 2 \binom{n+2}{i} (n+1-i) \\ &+ \sum_{i=1}^{n} 2 \binom{n+1}{i} \end{aligned}$$

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&+ \sum_{i=1}^{n} 2 \binom{n+2}{i} (n+1-i) \\
&+ \sum_{i=1}^{n} 2 \binom{n+1}{i} (n+1, i)
\end{aligned}$$

Collect Data.: 16, 80, 288, 896, 2560, 6912, 17920, 45056, 110592, 266240

Peak Sets

Def.

- The *peak set* of $w \in S_n$ is $\{i : w_{i-1} < w_i > w_{i+1}\}$.
- For $S \in \{1, 2, ..., n\}$, let $P(S; n)$ be the number of $w \in S_n$ with peak set S.
- Say S is n-admissible if $P(S; n) \neq 0$.

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Thm. (Billey-Burdzy-Sagan, last week/2012)
If $S = \{i_1 < \cdots < i_s\}$ is n-admissible then

$$\#P(S; n) = p(n)2^{n-s-1}$$

where $p(n)$ is a polynomial in n depending on S such that $p(n)$ is an integer for all integral n. Furthermore, $\deg p(n) = i_s - 1$.

When $S = \emptyset$ we have $\deg p(n) = 0$ and $\#P(S; n) = 2^{n-1}$.

Peak Sets

Cor. Given that $\deg p(n) = i_s - 1$, one can compute $p(n)$ explicitly given the first i_s non-zero values of $\#P(S;n)/2^{n-s-1}$ by the finite difference method.

Example. Consider $S = \{2, 4\}$. S is n-admissible for $n \geq 5$.

$$\begin{array}{r|c|c|c|c|c|c|} n = & 5 & 6 & 7 & 8 & 9 & 10 \\ \#P(S;n)/2^{n-s-1} = & 4 & 12 & 25 & 44 & 70 & 104 \end{array}$$

Take finite differences at least 3 times

$$\begin{array}{cccccc} 4 & 12 & 25 & 44 & 70 & 104 \\ 8 & 13 & 19 & 26 & 34 & \\ 5 & 6 & 7 & 8 & & \\ 1 & 1 & 1 & & & \\ 0 & 0 & & & & \end{array}$$

Therefore, $p_S(n) = 4\binom{n}{0} + 8\binom{n}{1} + 5\binom{n}{2} + \binom{n}{3}$.

Tenner's Database

In 2005, Bridget Tenner started collecting permutation patterns.

1. Database on finite lists of permutations with significance.
2. Permutations are independent of language, but not necessarily independent of notation and human variation. Makes it possible to find references and connections to other research.
3. Everyone can contribute!
4. Search by keywords or by patterns: e.g all containing w= 3 5 1 6 2 4

Learning Algorithms

Question. Given any subset of permutations, one can ask is the set closed under taking patterns? If so, how can we find the minimal permutation patterns that are missing?

Example.

$$\mathbf{(12)(21)}$$
$$\mathbf{(123)(213)(132)(312)(231)}$$
$$\mathbf{(1234)(2134)(1324)(3124)(231}$$
$$\mathbf{(1423)(4123)(2413)(1342)(3142)(3412)(2341)}$$

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Observe. Yes, set characterized by avoiding $(\mathbf{321})$ and all patterns of length 5.

Learning Algorithms

Classical pattern avoidance.:

Input: A Boolean function $f : S_\infty = \cup S_n \longrightarrow \{0, 1\}$.

Start: Set $B = \emptyset$.

Loop: For each $n = 1, 2, 3, \ldots N$
. Loop: For each $w \in S_n$
. if $f(w) \neq$ avoid-patterns(w, B) then
. Check: if $f(w) = 1$ then
. print "Not classical pattern avoidance!!"
. Else: $f(w) = 0$ so add w to B.

Marked Mesh Patterns

Def. (Bränden and Claesson) A *mesh pattern* is a permutation matrix with shaded entries.

Def. (Úlfarsson) A *marked mesh pattern* is a mesh pattern with numbers in the shaded regions.

Thm. (Henning Úlfarsson, 2012) The smooth, Gorenstein, factorial, defined by inclusions, 321-hexagon avoiding permutations can be described by marked mesh patterns.

See picture from “A Unification Of Permutation Patterns Related To Schubert Varieties” by Úlfarsson.

Open problems

1. Can someone add lci perms to Sloane's?
2. Can Lakshmibai's characterization of the tangent space basis for B,C,D be translated into signed patterns or a signed variation on marked mesh patterns.
3. What is the analog of marked mesh patterns for other types? Should be a statement about reflections in a Bruhat interval.
4. What is the Möbius function for the poset of pattern containment on S_∞?
5. What other types of theorems have canonical representations which might lead to more computer database tools? e.g. Hypergeometric series, integer sequences, patterns.

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Conclusion. Arigatai to omoimasu.