SCHUBERT POLYNOMIALS FOR THE CLASSICAL GROUPS

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1. Introduction

The task of a theory of *Schubert polynomials* is to produce explicit representatives for Schubert classes in the cohomology ring of a flag variety, and to do so in a manner that is as natural as possible from a combinatorial point of view.

To explain more fully, let us review a special case, the Schubert calculus for Grassmannians, where one asks for the number of linear spaces of given dimension satisfying certain geometric conditions. A typical problem is to find the number of lines meeting four given lines in general position in 3-space (answer below). For each of the four given lines, the set of lines meeting it is a *Schubert variety* in the Grassmannian and we want the number of intersection points of these four subvarieties.

In the modern solution of this problem, the Schubert varieties induce canonical elements of the cohomology ring of the Grassmannian, called *Schubert classes*. The product of these Schubert classes is the class of a point times the number of intersection points, counted with appropriate multiplicities. This reformulation of the problem, though one of the great achievements of algebraic geometry, is only part of a solution. It remains to give a concrete model for the cohomology ring that makes explicit computation with Schubert classes possible.

As it happens, the cohomology rings of Grassmannians can be identified with quotients of a polynomial ring so that Schubert classes correspond to Schur functions. Intersection numbers such as we are considering then turn out to be Littlewood–Richardson coefficients. For example, the answer to our four-lines problem is the coefficient of the Schur function $s_{(2,2)}$ in the product $s_{(1)}^4$, or 2. For an extended treatment and history of the subject, see [**10**], [**11**], [**18**].

The identification of Schur functions as Schubert polynomials for Grassmannians is a consequence of a more general and now highly developed theory of Schubert polynomials for the flag varieties of the special linear groups $SL(n, \mathbb{C})$. The starting point for this more general theory is a construction of Schubert classes by Bernstein–Gelfand–Gelfand [**1**] and

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Demazure [**4**]. Considering the flag variety of any simple complex Lie group, they show that beginning with a cohomology class of highest codimension (the Schubert class of a point), one obtains all Schubert classes by applying a succession of *divided difference* operators corresponding to simple roots.

At the level of computation in the cohomology ring, the preceding construction is quite explicit. It can be made even more explicit by selecting a specific polynomial to represent the top cohomology class. At first glance one sees no preferred way of making this choice. For the groups $SL(n,\mathbb{C})$, however, Lascoux and Schützenberger [**12**] made the crucial observation that one particular choice yields Schubert polynomials that represent the Schubert classes simultaneously for all n. This has led to a far-reaching theory with beautiful combinatorial ramifications, developed in [**3**], [**7**], [**13**].

Our goal in the present work is to replicate the theory of $SL(n,\mathbb{C})$ Schubert polynomials for the other infinite families of classical Lie groups and their flag varieties—the orthogonal groups $SO(2n,\mathbb{C})$ and $SO(2n+1,\mathbb{C})$ and the symplectic groups $Sp(2n,\mathbb{C})$. In principle, the program for doing this is simple. Within each family, the flag variety of order n embeds into the one of order $n+1$ as a Schubert variety, inducing maps on cohomology that send Schubert classes to Schubert classes. This set-up yields Schubert polynomials in the inverse limit, which can be calculated as the unique solution of an infinite system of divided difference equations.

In practice, there are delicate aspects to carrying out the program. The ring in which the Schubert polynomials lie must be correctly identified, and a particular change of variables made in order to express them in their natural form. The divided difference equations corresponding to generators of the symmetric group are easily dealt with, because our theory for $SL(n,\mathbb{C})$ reduces to the existing Lascoux–Schützenberger theory. The action of the 'extra' generators for the other Weyl groups is subtle, however, and their proper handling is the key to our calculations.

Ultimately, we solve the defining equations with polynomials defined by two equivalent formulas. One is an analog of the Billey–Jockusch–Stanley formula, while the other, which has no counterpart in the $SL(n)$ theory, expresses our polynomials in terms of $SL(n)$ Schubert polynomials and Schur Q- or P-functions. Our second formula involves the 'shifted Edelman–Greene correspondences' introduced in [**9**]. The mysterious 'third' correspondence found in that paper (and inappropriately called "type C" there) turns out to be the natural correspondence for the family D_n of even orthogonal groups.

Implicit in our results are new proofs of theorems of Pragacz [**15**] identifying the Schubert polynomials for *isotropic Grassmannians*—varieties consisting of subspaces isotropic with respect to a symplectic form on $\mathbb{C}^{2n}$ or a symmetric form on $\mathbb{C}^{2n}$ or $\mathbb{C}^{2n+1}$. Pragacz shows that these Schubert polynomials are Schur Q- or P-functions, in the same sense that the Schubert polynomials for ordinary Grassmannians are ordinary Schur functions.

Fomin and Kirillov [**6**] independently conjectured a formula almost identical to our Theorem 3, (2.5) for symplectic Schubert polynomials, based on considerations very different from ours. This suggests the possibility that these polynomials may be significant in as yet

unexpected contexts.

Logically speaking, geometry enters our work primarily as motivation, the proofs of the main theorems being entirely elementary in nature. The one exception for which we must appeal to a geometric proof is Theorem 5—the non-negativity of product expansion coefficients. In principle, the paper can be read and understood without any knowledge of Lie groups, flag varieties, or cohomology, if one is willing to accept on faith the *raison d'etre* for the construction.

2. Definitions, synopsis, and geometric underpinnings.

We begin this section with our definition of Schubert polynomials and a synopsis of the main theorems, followed by a review of the geometry needed to justify our definitions as correct and essentially inevitable, and to establish Theorem 5.

In Section 3 we introduce the combinatorial and symmetric function machinery we will need. Finally in Section 4, having formulated the problem as a system of divided difference equations, we prove directly that the solutions exist, are unique, are given by our formulas, and form an integral basis for the ring in which they lie.

Let $\mathbb{Q}[z_1, z_2, \dots]$ be the ring of polynomials in countably many variables z_i. Although the number of variables is infinite, a polynomial always has a finite number of terms. Let $p_k = z_1^k + z_2^k + \cdots$ denote the k-th power sum—a formal power series, not a polynomial. Then $\mathbb{Q}[z_1, z_2, \dots ; p_1, p_3, \dots]$ is the ring of formal power series which are polynomials in the z_i and the p_k (k odd). These are all algebraically independent, so $\mathbb{Q}[z_1, z_2, \dots ; p_1, p_3, \dots]$ can also be regarded simply as the polynomial ring in variables z_i and p_k.

The *infinite hyperoctahedral group* B_∞ is the union $\bigcup_{n=1}^{\infty} B_n$, where B_n is the group of signed permutations on $\{1, \dots, n\}$. Here we regard B_m as a subgroup of B_n for $m < n$ in the obvious way. B_∞ is generated by the transpositions $\sigma_i = (i\ i{+}1)$ and the sign change $\sigma_0(1) = \overline{1}$. We make B_∞ act on the formal power series ring $\mathbb{Q}[\![z_1, z_2, \dots]\!]$ by letting σ_i interchange z_i and z_{i+1}, for $i \geq 1$, and letting σ_0 replace z_1 with $-z_1$. Restricting this action to the rings $\mathbb{Q}[z_1, z_2, \dots]$ and $\mathbb{Q}[z_1, z_2, \dots ; p_1, p_3, \dots]$, we see that the σ_i act as described on the z_i's, that σ_i fixes the p_k's for $i \geq 1$, and that $\sigma_0 p_k = p_k - 2z_1^k$.

The symmetric groups S_n and their union S_∞ act as subgroups of B_∞ generated by the transpositions σ_i for $i \geq 1$.

The group D_n is the subgroup of elements $w \in B_n$ which make an even number of sign changes. Their union we denote D_∞. The standard generators for these groups are σ_i for $i \geq 1$ and an additional generator $\sigma_{\hat{1}} = \sigma_0 \sigma_1 \sigma_0$ which replaces z_1 with $-z_2$ and z_2 with $-z_1$.

On the rings $\mathbb{Q}[z_1, z_2, \dots]$ and $\mathbb{Q}[z_1, z_2, \dots ; p_1, p_3, \dots]$ we define *divided difference* opera-

tors

$$\begin{aligned}
\partial_i f &= \frac{f - \sigma_i f}{z_i - z_{i+1}} \quad (i \geq 1), \\
\partial_0 f &= \frac{f - \sigma_0 f}{-2z_1}, \\
\partial_0^B f &= \frac{f - \sigma_0 f}{-z_1}, \\
\partial_{\hat{1}} f &= \frac{f - \sigma_{\hat{1}} f}{-z_1 - z_2}.
\end{aligned} \tag{2.1}$$

The denominator in each case is a linear form defining the hyperplane fixed by the reflection σ_i, so these fractions are actually polynomials.

Definition. The *Schubert polynomials of type A* are elements $\mathfrak{S}_w \in \mathbb{Q}[z_1, z_2, \dots]$ for $w \in S_\infty$ satisfying the equations

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{w\sigma_i} & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w), \end{cases} \tag{2.2}$$

for all $i \geq 1$, together with the condition that the constant term of $\mathfrak{S}_w$ is 1 if $w = 1$ and 0 otherwise.

Definition. The *Schubert polynomials of type C* are elements $\mathfrak{C}_w \in \mathbb{Q}[z_1, z_2, \dots ; p_1, p_3, \dots]$ for $w \in B_\infty$ satisfying the equations

$$\partial_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{w\sigma_i}, & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w), \end{cases} \tag{2.3}$$

for all $i \geq 0$, together with the condition that the constant term of $\mathfrak{C}_w$ is 1 if $w = 1$ and 0 otherwise. The *Schubert polynomials of type B* are $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$, where $s(w)$ is the number of signs changed by w. These satisfy equations (2.3) with ∂_0^B in place of ∂_0.

Definition. The *Schubert polynomials of type D* are elements $\mathfrak{D}_w \in \mathbb{Q}[z_1, z_2, \dots ; p_1, p_3, \dots]$ for $w \in D_\infty$ satisfying the equations

$$\partial_i \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{w\sigma_i}, & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w), \end{cases} \tag{2.4}$$

for all $i \geq 1$ and $i = \hat{1}$, together with the condition that the constant term of $\mathfrak{D}_w$ is 1 if $w = 1$ and 0 otherwise.

In the above definitions $l(w)$ refers to the minimum length of an expression for w as a product of the simple reflections σ_i generating the given group. This is well-defined in S_∞, B_∞, and D_∞, since length of w in the Coxeter group S_n, B_n, or D_n is independent of n.

Theorem 1. *Solutions of the defining equations for each type of Schubert polynomial exist and are unique.*

Theorem 2. *The Schubert polynomials of type A are the same as those defined by Lascoux–Schützenberger* [**12**].

To state our main results in their most attractive form, we replace the ring $\mathbb{Q}[z_1, z_2, \dots ; p_1, p_3, \dots]$ with the isomorphic ring $\mathbb{Q}[z_1, z_2, \dots ; p_1(X), p_3(X), \dots]$, where $p_k(X) = x_1^k + x_2^k + \cdots$ are power sums in new variables, and we identify $p_k(X)$ with $-p_k(Z)/2$.

Recall (or see Section 3) that the Schur Q-functions $Q_\mu(X)$, for μ a partition with distinct parts, form a basis for the subring $\mathbb{Q}[p_1(X), p_3(X), \dots]$ of the ring of symmetric functions in the variables X. One also defines $P_\mu(X) = 2^{-l(\mu)} Q_\mu(X)$. Then the subrings $\mathbb{Z}[Q_\mu(X)]$ and $\mathbb{Z}[P_\mu(X)]$ have integral bases $\{Q_\mu(X)\}$ and $\{P_\mu(X)\}$ respectively.

Theorem 3. *The Schubert polynomials $\mathfrak{C}_w$ are given by*

$$\mathfrak{C}_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v \in S_\infty}} F_u(X) \mathfrak{S}_v(Z), \tag{2.5}$$

where $F_u(X)$ is a certain non-negative integral linear combination of Schur Q-functions computed from $u \in B_\infty$ via the B_n Edelman–Greene correspondence (see Section 3). The polynomials $\mathfrak{C}_w$ are a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[z_1, z_2, \dots ; Q_\mu(X)]$. The polynomials $\mathfrak{B}_w = 2^{-s(w)} \mathfrak{C}_w$ are a $\mathbb{Z}$-basis for $\mathbb{Z}[z_1, z_2, \dots ; P_\mu(X)]$. Given a partition $\mu = (\mu_1 > \mu_2 > \cdots > \mu_l)$ with distinct parts, let

$$w = \overline{\mu_1}\, \overline{\mu_2} \dots \overline{\mu_l} 12 \dots . \tag{2.6}$$

Then we have

$$\mathfrak{C}_w = Q_\mu(X), \quad \mathfrak{B}_w = P_\mu(X). \tag{2.7}$$

In (2.6) the bars denote minus signs, and the ellipsis at the end stands for the remaining positive integers, omitting the μ_i's, in increasing order. These notational conventions for signed permutations will be used below without further comment.

Theorem 4. *The Schubert polynomials $\mathfrak{D}_w$ are given by*

$$\mathfrak{D}_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v \in S_\infty}} E_u(X) \mathfrak{S}_v(Z), \tag{2.8}$$

where $E_u(X)$ is a certain non-negative integral linear combination of Schur P-functions computed from $u \in D_\infty$ via the D_n Edelman–Greene correspondence (see Section 3). The polynomials $\mathfrak{D}_w$ are a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[z_1, z_2, \dots ; P_\mu(X)]$. Given a partition $\mu = (\mu_1 >$

$\mu_2 > \cdots > \mu_l$) with distinct parts, let $\nu_i = 1 + \mu_i$, taking $\mu_l = 0$ if necessary to make the number of parts even. Then for

$$w = \overline{\nu_1}\,\overline{\nu_2}\ldots\overline{\nu_l}12\ldots, \tag{2.9}$$

we have

$$\mathfrak{D}_w = P_\mu(X). \tag{2.10}$$

Further formulas expressing $\mathfrak{B}_w$, $\mathfrak{C}_w$, and $\mathfrak{D}_w$ as sums over admissible monomials for reduced decompositions of w are given in Section 4, where they play a crucial role in the proofs of the theorems.

Theorem 5. *In the product expansions*

$$\mathfrak{S}_u\mathfrak{S}_v = \sum_w c_{uv}^w \mathfrak{S}_w \tag{2.11}$$

and like expansions for types B, C, and D, the coefficients c_{uv}^w are non-negative.

Proof. For n large enough, we have a homomorphism sending the Schubert polynomials appearing here to Schubert classes. The coefficient c_{uv}^w is therefore the same as that appearing in (2.12) below. □

We now review the facts we require about flag varieties and their cohomology rings. The goal is to discover a uniform definition of Schubert polynomials for the classical families, and finally to specialize it to obtain our definitions (2.1)–(2.4) of Schubert polynomials of types A, B, C and D. Along the way we fix notation for the classical groups, describe their flag varieties concretely, and state enough properties of Schubert classes to justify Theorem 5. The general theory applies to any connected and simply connected semisimple complex Lie group G, but we make things fully explicit only for the classical families. Primary references are [**1**], [**2**].

Let B be a Borel subgroup of G. Abstractly, the *flag variety* is the space of cosets $X = G/B = \{gB \mid g \in G\}$, which is a smooth complex projective variety. For the classical groups, the flag varieties have the following more concrete descriptions.

For type A_{n-1}, $G = SL(n,\mathbb{C})$ acts naturally on $\mathbb{C}^n$. The Borel subgroup B of upper triangular matrices can be described as the stabilizer of the sequence of subspaces $E = (0 \subset \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_{n-1}\rangle \subset \mathbb{C}^n)$, where e_i are the unit coordinate vectors. Hence we may identify $X = G/B$ with the variety of *flags of subspaces* $F = (0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n)$ satisfying $\dim F_d = d$.

For types B_n ($G = SO(2n+1,\mathbb{C})$), C_n ($G = Sp(2n,\mathbb{C})$), and D_n ($G = SO(2n,\mathbb{C})$), G is by definition the group of automorphisms preserving a non-degenerate bilinear form $(-,-)$ on $V = \mathbb{C}^{2n}$ or $\mathbb{C}^{2n+1}$. For SO, this will be a symmetric form; for Sp, a skew form $(x,y) = -(y,x)$. To be definite, let us agree that the matrix $[(e_i,e_j)]_{i,j}$ of the form on unit coordinate vectors shall be J_{2n} or J_{2n+1} for SO, and $\left[\begin{smallmatrix} 0 & J_n \\ -J_n & 0 \end{smallmatrix}\right]$ for Sp, where J_l is the $l \times l$ 'reverse identity matrix' with entries 1 on the anti-diagonal and 0 elsewhere. With these

choices, the upper triangular matrices in G form a Borel subgroup in each case, which we take as B.

A subspace $W \subseteq V$ is *isotropic* if $(x, y) = 0$ for all $x, y \in W$. An *isotropic flag* is a partial flag $F = (0 \subset F_1 \subset F_2 \subset \cdots \subset F_n)$ with $\dim F_d = d$ and F_n isotropic. (Note that maximal isotropic subspaces have dimension n.) Letting $F_i^\perp$ denote the orthogonal complement of F_i, we have $F_n^\perp \subset F_{n-1}^\perp \subset \cdots \subset F_1^\perp \subset V$ and $F_n \subseteq F_n^\perp$ (in fact $F_n = F_n^\perp$ except for type B_n). By adding the orthogonal complements $F_i^\perp$, we thus extend each isotropic flag to a complete flag, which of course has the same stabilizer in G. In particular, B is the stabilizer of the isotropic flag $E = (0 \subset \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_n\rangle)$. Hence we identify $X = G/B$ with the variety of isotropic flags. (Except for type D_n, where the space of isotropic flags has two components, corresponding to the two cosets of $SO(2n, \mathbb{C})$ in $O(2n, \mathbb{C})$, and we identify X with the component containing E).

The B-orbits in X correspond to double cosets BgB in G. By the Bruhat decomposition $G = \bigcup_{w\in W} BwB$, the flags wE for w in the Weyl group W form a system of representatives for these orbits. The orbit $X_w = BwE \subseteq X$ is isomorphic to the affine cell $\mathbb{C}^{l(w)}$, where $l(w)$ denotes length of w. These cells are called *Schubert cells* and their closures $\overline{X}_w$ *Schubert varieties*. Since $\overline{X}_w$ is B-stable, it is a union of B-orbits X_v.

For each $w \in W$, the fundamental homology class of the subvariety $\overline{X}_w \subseteq X$ determines an element $[\overline{X}_w] \in H_{2l(w)}(X, \mathbb{Z})$ and the decomposition of X into Schubert cells implies that the elements $[\overline{X}_w]$ form a $\mathbb{Z}$-basis of $H_*(X, \mathbb{Z})$. The *Schubert classes* $C_w \in H^{2l(w)}(X, \mathbb{Q})$ are the dual basis defined by $\langle C_w, [\overline{X}_v]\rangle = \delta_{v,w}$, where $\langle -, -\rangle$ is the natural pairing of cohomology and homology.

Let w_0 denote the longest element of W. Then by [**1**], Corollary 3.19, we have $C_{w_0 w}C_w = C_{w_0}$, and $C_v C_w = 0$ for $l(v) + l(w) = l(w_0)$ and $v \neq w$. Thus C_w is the cohomology class associated via Poincaré duality with the algebraic cycle $\overline{X}_{w_0 w}$; in other words $[\overline{X}_{w_0 w}]$ is the cap product of C_w with $[\overline{X}_{w_0}]$.

By intersection theory it follows that the coefficients c_{uv}^w defined for $l(u) + l(v) = l(w)$ by setting in $H^*(X, \mathbb{Q})$

$$C_u C_v = \sum_{w\in W} c_{uv}^w C_w, \quad \text{or equivalently} \quad C_u C_v C_{w_0 w} = c_{uv}^w C_{w_0}, \tag{2.12}$$

are non-negative integers. Namely, c_{uv}^w is the intersection number of cycles equivalent to $\overline{X}_{w_0 u}$, $\overline{X}_{w_0 v}$, and $\overline{X}_w$ in general position.

As in [**1**], Proposition 1.3, we identify the rational cohomology ring $H^*(X)$ with the quotient $\mathbb{Q}[P]/I$ of the symmetric algebra of the root space P by the ideal I generated by all non-constant homogeneous W-invariant polynomials. For each simple root α and corresponding reflection σ_α, we then define the divided difference operator ∂_α on $\mathbb{Q}[P]$ as in [**1**] by

$$\partial_\alpha f = \frac{f - \sigma_\alpha f}{\alpha}. \tag{2.13}$$

If $\sigma_\alpha g = g$ then clearly $\partial_\alpha(gf) = g\partial_\alpha f$, which shows that the operators ∂_α carry the ideal I into itself, and therefore induce well-defined operators ∂_α on $H^*(X)$. Theorem 3.14 of [**1**] states that the Schubert classes C_w satisfy

$$\partial_\alpha C_w = \begin{cases} C_{w\sigma_\alpha} & \text{if } l(w\sigma_\alpha) < l(w) \\ 0 & \text{if } l(w\sigma_\alpha) > l(w). \end{cases} \tag{2.14}$$

These equations, together with their dimensions and the fact that $C_1 = 1$, determine the classes C_w.

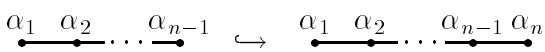
Embeddings of flag varieties, root systems, and Weyl groups arise when the Dynkin diagram of one group is a subdiagram of another. Let D be a Dynkin diagram of finite type and $\mathfrak{g}$ the semisimple complex Lie algebra constructed from it, with Cartan subalgebra $\mathfrak{h}$, Borel subalgebra $\mathfrak{b}$, root system Δ and basis of simple roots Σ. Let $H \subseteq B \subseteq G$ be the corresponding connected and simply connected semisimple complex Lie group with maximal torus H and Borel subgroup B. Given a subset $S \subseteq \Sigma$ of the simple roots let $D' \subseteq D$ be the subdiagram whose vertices correspond to the elements of S. We then have maps $\phi\colon G' \hookrightarrow G$ and $d\phi\colon \mathfrak{g}' \hookrightarrow \mathfrak{g}$, with $\phi(B') \subseteq B$ and $\phi(H') \subseteq H$.

Dual to $d\phi$ we have a map $\pi\colon P \to P'$ between the root spaces. The map π carries the subspace $\mathbb{Q}\langle S\rangle \subseteq P$ spanned by S onto P' and sends $(\Delta \cap \mathbb{Q}\langle S\rangle, S)$ isomorphically onto the root system (Δ', Σ'). Identifying the Weyl group W' with the subgroup of W generated by $\{\sigma_\alpha \mid \alpha \in S\}$, π is W'-equivariant.

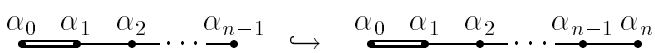
The isomorphisms $\mathbb{Q}[P]/I \cong H^*(G/B)$ of [**1**] commute with the maps $\tilde{\phi}^*\colon H^*(G/B) \to H^*(G'/B')$ induced by $\tilde{\phi}\colon G'/B' \to G/B$ and $\tilde{\pi}\colon \mathbb{Q}[P]/I \to \mathbb{Q}[P']/I'$ induced by $\pi\colon P \to P'$. Letting ∂_α, ∂'_α denote respectively the divided difference operators on $\mathbb{Q}[P]/I$, $\mathbb{Q}[P']/I'$ corresponding to a root $\alpha \in S$, we have $\tilde{\pi}\partial_\alpha = \partial'_\alpha\tilde{\pi}$. Hence by the remark following (2.14), $\tilde{\phi}^*(C_w) = C'_w$, where C_w and C'_w denote the respective Schubert classes in $H^*(G/B)$ and $H^*(G'/B')$ indexed by $w \in W'$.

For each classical family, let G_n denote the n-th group—$SL(n)$, $SO(2n+1)$, $Sp(2n)$, or $SO(2n)$. We define maps $\phi_n : G_n \hookrightarrow G_{n+1}$ from Dynkin diagram embeddings as shown below.

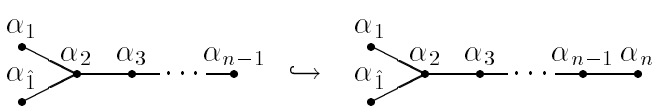
(2.15) Type A :



(2.16) Types B,C :



(2.17) Type D :



In terms of the explicit matrix representations described earlier the maps ϕ_n take the

forms

$$\phi_n(M) = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array}\right] \text{ for type } A; \qquad \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & M & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \text{ for types } B,\ C,\ D. \tag{2.18}$$

For type A, $G = SL(n)$, we express the elements of H—the diagonal matrices—in the form

$$\exp \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}, \quad \text{where } z_1 + \cdots + z_n = 0. \tag{2.19}$$

Then the root space is $P_n = \mathbb{Q}\langle z_1, \dots, z_n\rangle / \mathbb{Q}\langle z_1 + \cdots + z_n\rangle$, the Weyl group W is S_n, and the cohomology ring is $\mathbb{Q}[P_n]/I_n = \mathbb{Q}[z_1, \dots, z_n]/(p_1, p_2, \dots)$ where $p_k = z_1^k + \cdots + z_n^k$. From (2.18) we see that the map $\tilde{\pi}\colon Q[z_1, \dots, z_{n+1}]/(p_1, p_2, \dots) \to Q[z_1, \dots, z_n]/(p_1, p_2, \dots)$ is given by sending z_{n+1} to zero. The simple roots are $\alpha_i = z_i - z_{i+1}$.

For types B, C and D, $G = SO(2n+1)$, $Sp(2n)$ or $SO(2n)$, we express elements of H in the form

(2.20)

$$\exp \begin{bmatrix} -z_n & & & & & & \\ & \ddots & & & & & \\ & & -z_1 & & & & \\ & & & 0 & & & \\ & & & & z_1 & & \\ & & & & & \ddots & \\ & & & & & & z_n \end{bmatrix} \text{ for } SO(2n+1), \text{ or } \exp \begin{bmatrix} -z_n & & & & \\ & \ddots & & & \\ & & -z_1 & & \\ & & & z_1 & \\ & & & & \ddots \\ & & & & & z_n \end{bmatrix} \text{ for } Sp(2n),\ SO(2n).$$

In each case we then have $P_n = \mathbb{Q}\langle z_1, \dots, z_n\rangle$, $W = B_n$ for types B, C or D_n for type D, $\mathbb{Q}[P_n]/I_n = \mathbb{Q}[z_1, \dots, z_n]/(p_2, p_4, \dots)$ for types B and C, modulo an additional invariant $z_1 \cdots z_n \in I_n$ for type D, and $\tilde{\pi}\colon \mathbb{Q}[P_{n+1}]/I_{n+1} \to \mathbb{Q}[P_n]/I_n$ given by sending z_{n+1} to zero. The simple roots are $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ for types B, C, and $\alpha_{\hat{1}}, \alpha_1, \dots, \alpha_{n-1}$ for type D, where $\alpha_0 = -z_1$ for type C, $\alpha_0 = -2z_1$ for type B, $\alpha_{\hat{1}} = -z_1 - z_2$, and $\alpha_i = z_i - z_{i+1}$ for $i = 1, 2, \dots$ as before.

Now we come to the general definition of Schubert polynomials.

Definition. For any of the classical families A, B, C, or D, let the n-th group be G_n, with Weyl group W_n and flag variety X_n. Let $W_\infty = \varinjlim W_n$ be the direct limit of the Weyl groups. For $w \in W_\infty$ the *Schubert polynomial* S_w is the element $\varprojlim C_w$ in the inverse limit $\varprojlim H^*(X_n)$ of the system

$$\cdots \leftarrow H^*(X_n) \leftarrow H^*(X_{n+1}) \leftarrow \cdots . \tag{2.21}$$

In the explicit coordinates (2.19)–(2.20) the general divided difference equations (2.13)–(2.14) clearly specialize to our definitions (2.1)–(2.4). To fully justify the original definitions we have only to verify that the Schubert polynomials, defined now as elements of the inverse limit $\varprojlim \mathbb{Q}[P_n]/I_n$, belong to the ring $R = \mathbb{Q}[z_1, z_2, \dots]$ (type A) or

$R = \mathbb{Q}[z_1, z_2, \ldots ; p_1, p_3, \ldots]$ (types B, C, D). Strictly speaking, we mean that the S_w belong to the image of R under the map $R \hookrightarrow \varprojlim \mathbb{Q}[P_n]/I_n$ induced by the obvious maps $R \to \mathbb{Q}[P_n]/I_n$.

Given $w \in W_\infty$, let N be large enough so that $w \in W_N$. Then the equations $\partial_i S_w = 0$ for $i > N$ imply that S_w is symmetric in all the variables $z_{N+1}, z_{N+2}, \ldots$. Since S_w is also homogeneous of degree $l(w)$, its image in $\mathbb{Q}[P_n]/I_n$ lies in the image of the subring $R' = \mathbb{Q}[z_1, \ldots, z_N, p_1, \ldots, p_{l(w)}]$. But the image of R' in $\mathbb{Q}[P_n]/I_n$ is constant for $n > N$, so the inverse limit of this subsystem is the image of R' in $\varprojlim \mathbb{Q}[P_n]/I_n$. This shows $S_w \in R$.

Finally, a few words on how our results generalize those of Pragacz [**15**] on isotropic Grassmannians. Projecting each isotropic flag $(0 \subset F_1 \subset F_2 \subset \cdots \subset F_n)$ on its last component F_n, we map the flag variety of type B, C, or D onto the corresponding isotropic Grassmannian, or variety of maximal isotropic subspaces with respect to the given symmetric or symplectic form. In group-theoretic terms, this projection is the natural map from G/B onto G/P, where the parabolic subgroup P is the stabilizer of the space E_n in the base flag E. The Schubert varieties in the Grassmannian are, by definition, the closures of B-orbits.

The map $X = G/B \to G/P$ induces a map embedding the cohomology ring of the isotropic Grassmannian into $H^*(X)$ as the subring of W_P-invariants, where W_P is the "parabolic" subgroup of the Weyl group for which $P = W_P B$. The embedding sends Schubert classes to Schubert classes, so we may identify the Schubert polynomials for G/P with those Schubert polynomials for X which are W_P invariant (see [**1**] for details).

For isotropic Grassmanians, W_P is S_n, so the relevant Schubert polynomials $\mathfrak{B}_w$, $\mathfrak{C}_w$, or $\mathfrak{D}_w$ are the ones indexed by signed permutations w with $w(i) < w(i+1)$ for all i. Of course, such a w has all its negative values $w(i)$ at the beginning, in decreasing order of absolute value, followed by all the positive values in increasing order. In short, it has the form of equation (2.6) or (2.9), and the corresponding Schubert polynomial is therefore a Schur Q- or P-function, according to Theorem 3 or 4. Thus we recover the theorems of Pragacz.

3. The shifted Edelman–Greene correspondence

In this section we review the shifted Edelman–Greene correspondences from [**9**] and use them to define symmetric functions associated with elements of the Weyl groups B_n and D_n. These symmetric functions are the natural B_n and D_n analogs of symmetric functions defined for elements of A_n by Stanley [**17**]. For this reason we call them *Stanley functions*. Just as the A_n Stanley functions are now understood to be 'stable' type A Schubert polynomials [**13**], the B_n and D_n Stanley functions turn out to be specializations of type B and D Schubert polynomials.

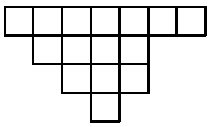
Our central results here are identities between the defining *tableau* forms of the Stanley functions and more explicit *monomial* forms given by Propositions 3.4 and 3.10. Expressed in tableau form, the Stanley functions are transparently non-negative integral combinations of Schur Q- and P- functions, respectively. Expressed in monomial form, they are amenable to detailed computations with divided difference operators. Both aspects are essential for

the proofs of our main theorems in Section 4.

At the end of this section we evaluate the Stanley functions for various special elements of the Weyl groups. Most of these evaluations and some others not given here were also found by J. Stembridge, T.-K. Lam, or both, in work not yet published. They take the monomial forms as the definition, attributing this to Fomin. We give a self-contained treatment here, since our methods are new and the proofs simple. Note, however, that Propositions 3.13 and 3.14 were first proved by Stembridge, and Proposition 3.16 by Lam. They consider Proposition 3.17 to be well-known!

We begin by reviewing the combinatorial definition of Schur Q- and P-functions. Recall that when $\mu = (\mu_1 > \mu_2 > \cdots > \mu_l)$ is a partition with distinct parts, the corresponding *shifted shape* is a sort of Ferrers diagram of μ, but with each row indented one space at the left from the preceding row, as shown here for $\mu = (7, 4, 3, 1)$.

(3.1)



A *tableau* of shape μ is a function assigning to each cell in the shape an entry from some totally ordered alphabet, so that the entries are non-decreasing along each row and column. If the alphabet is the set of numbers $\{1, \ldots, n\}$, where $n = |\mu|$, and the assignment of numbers to cells is bijective, the tableau is a *standard tableau*. If the alphabet consists of numbers $1, 2, \ldots$ and circled numbers $1^\circ, 2^\circ, \ldots$, with the ordering $1^\circ < 1 < 2^\circ < 2 < \cdots$, the tableau is a *circled tableau* provided that no circled number is repeated in any row and no uncircled number is repeated in any column.

If T is a circled tableau, its *weight* is the monomial x^T in variables $x_1, x_2, \ldots$ formed by taking the product over all entries in T of the variable x_i for an entry i or i°.

Definition. The *Schur Q-function* $Q_\mu(X)$ is the sum $\sum_T x^T$, taken over all circled tableaux of shifted shape μ. The *Schur P-function* $P_\mu(X)$ is defined to be $2^{-l(\mu)} Q_\mu(X)$, where $l(\mu)$ is the number of parts of μ.

Note that the rules defining circled tableaux always permit free choice of the circling for entries along the main diagonal. Consequently, P_μ can also be described as the sum $\sum_T x^T$, taken over circled tableaux with no circled entries on the diagonal.

The following well-known basic facts can be derived (albeit with some effort) from various theorems and exercises in [**14**].

Proposition 3.1. *The Schur P- and Q-functions are the specializations $P_\mu(X; -1)$ and $Q_\mu(X; -1)$ of the Hall-Littlewood polynomials $P_\mu(X; t)$ and $Q_\mu(X; t)$, for μ with distinct parts. Consequently, they are symmetric functions in the variables X and they depend only upon the power sums $p_k(X)$ for k odd. Moreover, the sets $\{P_\mu(X)\}$ and $\{Q_\mu(X)\}$ are $\mathbb{Q}$-bases for the algebra $\mathbb{Q}[p_1(X), p_3(X), \ldots]$ generated by odd power sums, and $\mathbb{Z}$-bases for the subrings $\mathbb{Z}[P_\mu]$ and $\mathbb{Z}[Q_\mu]$, respectively.*

For more detail on the combinatorial interpretation of P- and Q-functions, consult [**16**], [**20**].

Next we need a description of Q-functions in terms of standard tableaux. If T is a (shifted) standard tableau of size n, we say that $j \in \{1, \ldots, n-1\}$ is a *descent* of T if $j+1$ appears in a lower row than j in T. The set of descents is denoted $D(T)$. We say that $j \in \{2, \ldots, n-1\}$ is a *peak* of T if $j-1$ is an ascent and j is a descent. The set of peaks we denote $P(T)$.

Given a set $P \subseteq \{2, \ldots, n-1\}$ (to be thought of as a peak set), we say that a sequence $i_1 \leq i_2 \leq \cdots \leq i_n$ is *P-admissible* if we do not have $i_{j-1} = i_j = i_{j+1}$ for any $j \in P$. Letting $A(P)$ denote the set of P-admissible sequences, we define the *shifted quasi-symmetric function*

$$\Theta_P^n(X) = \sum_{\substack{(i_1 \leq \cdots \leq i_n) \\ \in A(P)}} 2^{|\mathbf{i}|} x_{i_1} x_{i_2} \cdots x_{i_n}, \tag{3.2}$$

where $|\mathbf{i}|$ denotes the number of *distinct* values i_j in the admissible sequence, *i.e.*, the number of distinct variables in the monomial.

Proposition 3.2. *The Schur Q-function Q_μ is equal to the sum $\sum_T \Theta_{P(T)}^{|\mu|}(X)$, where T ranges over standard tableau of shifted shape μ.*

Proof. Our argument is a routine one, involving subscripting the entries of each circled tableau to get a standard tableau, so we only sketch it. A similar proof of the analogous formula for Schur S-functions originated in unpublished work of I. Gessel.

Given a circled tableau T, all entries i° and i for any given i form a rim hook, not necessarily connected, with the i°'s occupying the vertical portions and the i's occupying the horizontals. To obtain an underlying standard tableau, we distinguish all occurrences of i° by subscripts $i_1^\circ, i_2^\circ, \ldots$, proceeding downward by rows. In a similar fashion we distinguish occurrences of i proceeding to the right by columns. By this subscripting we totally order all entries of T; replacing them by the numbers 1 through $n = |T|$ in the same order gives a standard tableau $S(T)$.

Given $S(T)$ and the weight monomial x^T, we immediately recover T, except for the circling. The entries of $S(T)$ corresponding to i° and i form a sequence which descends and then ascends, *i.e.*, a sequence with no peak. Henceforth we refer to such a sequence as a *vee*. We must have i° along the descending part of the vee and i along the ascending part. Only the circling at the 'valley' of the vee is undetermined. Thus there are $2^{|\mathbf{i}|}$ circled tableaux with this particular weight and underlying standard tableau, where $|\mathbf{i}|$ is the number of distinct indices in the weight monomial. Moreover, the combinations of standard tableau S and weight monomial $x_{i_1} x_{i_2} \cdots x_{i_n}$ that occur are exactly those where the sequence $i_1 \leq \cdots \leq i_n$ is admissible for the peak set $P(S)$. This proves the proposition. □

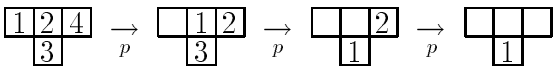
Having completed our review of Q- and P-functions, we turn to the Edelman–Greene correspondences and associated Stanley functions. We treat B_n first since everything we need is proven in [**9**]; for D_n we will have to add something.

Definition. A *reduced word* for an element $w \in B_n$ is a sequence $\mathbf{a} = a_1 a_2 \dots a_l$ of indices $0 \le a_i \le n-1$ such that w is the product of simple reflections $\sigma_{a_1} \cdots \sigma_{a_l}$ and $l = l(w)$ is minimal. We denote by $R(w)$ the set of reduced words for w. The *peak set* $P(\mathbf{a})$ is the set $\{i \in \{2, \dots, l-1\} \mid a_{i-1} < a_i > a_{i+1}\}$.

Definition. Let $\boldsymbol{\beta}_n$ denote the shifted 'staircase' shape $(2n-1, 2n-3, \dots, 1)$ of size n^2. Let its corners be lapelled $0, 1, \dots, n-1$ from the bottom row to the top. If T is a standard tableau of shape $\boldsymbol{\beta}_n$, its *promotion sequence* $\hat{p}(T)$ is the sequence $a_1 \dots a_{n^2}$ in which a_i is the label of the corner occupied by the largest entry of $p^{n^2-i}(T)$. Here the *promotion operator* p is defined as follows: to compute $p(T)$, delete the largest entry of T, perform a (shifted) *jeu-de-taquin* slide into its cell, and fill the vacated upper-left corner with a new least entry.

Since this definition is a bit complicated, we illustrate with a simple example. Taking $n = 2$, let T be the first tableau pictured below. Its promotions $p(T)$, $p^2(T)$, $p^3(T)$ are shown to its right, except we have suppressed the new entries that should fill the upper left.

(3.3)



Each a_i is 0 or 1, according to which corner is occupied by the largest entry of $p^{4-i}(T)$. Note that the largest entry of $p^{4-i}(T)$ is i itself, so a_i records the corner ultimately reached by entry i in the promotion process. Here the sequence $\mathbf{a} = \hat{p}(T)$ is 0101.

Proposition 3.3. (B_n Edelman–Greene correspondence) *The map $T \mapsto \hat{p}(T)$ is a bijection from standard tableaux of shape $\boldsymbol{\beta}_n$ to reduced words for the longest element $w_0 = \overline{1}\,\overline{2} \cdots \overline{n}$ of B_n. The initial segment $a_1 \dots a_k$ of the reduced word $\hat{p}(T)$ determines the initial segment $T|_k$ containing entries 1 through k of T. Denoting $T|_k$ by $\Gamma(a_1 \dots a_k)$, the number*

$$f_w^\mu = |\{\mathbf{a} \in R(w) \mid \Gamma(\mathbf{a}) = S\}| \tag{3.4}$$

depends only on w and on the shape μ of S. Finally, we have $P(\mathbf{a}) = P(\Gamma(\mathbf{a}))$ for the peak sets.

Proof. All but the part about peak sets is proved in Proposition 6.1 and Theorem 6.3 of [**9**]. For the peak set part it suffices to show $P(T) = P(\hat{p}(T))$ for T of shape $\boldsymbol{\beta}_n$. For a peak at position $n^2 - 1$, that is, involving the largest three entries of T, it is obvious that T has a peak if and only if $\hat{p}(T)$ does. For other positions, the result follows because shifted *jeu-de-taquin* preserves the peak set of a tableau. □

Using Proposition 3.3 we can now introduce well-defined symmetric functions associated with elements of B_n.

Definition. Let w be an element of B_n. The B_n *Stanley function* $F_w(X)$ is defined by

$$F_w(X) = \sum_\mu f_w^\mu Q_\mu(X). \tag{3.5}$$

The following crucial identity is an immediate consequence of Propositions 3.2 and 3.3.

Proposition 3.4.

$$\begin{aligned} F_w(X) &= \sum_{\mathbf{a}\in R(w)} \Theta^{l(w)}_{P(\mathbf{a})}(X) \\ &= \sum_{\mathbf{a}\in R(w)} \sum_{\substack{(i_1\le\cdots\le i_l)\\ \in A(P(\mathbf{a}))}} 2^{|\mathbf{i}|} x_{i_1} x_{i_2} \cdots x_{i_l}. \end{aligned} \tag{3.6}$$

From (3.6) we obtain another important identity.

Corollary 3.5. *For all w, $F_w(X) = F_{w^{-1}}(X)$.*

Proof. Since F_w is a symmetric function, it is unaltered by reversing the indices of the variables. Therefore (3.6) is equal to

$$\sum_{\mathbf{a}\in R(w)} \sum_{\substack{(i_1\ge\cdots\ge i_l)\\ \in A(P(\mathbf{a}))}} 2^{|\mathbf{i}|} x_{i_1} x_{i_2} \cdots x_{i_l}, \tag{3.7}$$

where the admissibility condition on a decreasing sequence is just as before: no $i_{j-1} = i_j = i_{j+1}$ when j is a peak. But then $(i_1 \ge \cdots \ge i_l)$ is admissible for $P(\mathbf{a})$ if and only if the reversed sequence $(i_l \le \cdots \le i_1)$ is admissible for $P(\mathbf{a}^r)$, where $\mathbf{a}^r$ is the reverse of $\mathbf{a}$, *i.e.*, a general element of $R(w^{-1})$. So (3.7) reduces to (3.6) for w^{-1}. □

The situation for D_n is analogous to that for B_n, but requires some new information about the relevant Edelman–Greene correspondence.

Definition. A *reduced word* for $w \in D_n$ is a sequence $a_1 a_2 \dots a_l$ of the symbols $\hat{1}, 1, 2, \dots, n-1$ such that $\sigma_{a_1} \cdots \sigma_{a_l} = w$ and $l = l(w)$ is minimal. As before, $R(w)$ denotes the set of reduced words for w. A *flattened word* is a word obtained from a D_n reduced word by changing all the $\hat{1}$'s to 1's. The *peak set* $P(\mathbf{a})$ is defined to be the peak set (in the obvious sense) of the corresponding flattened word. A *winnowed word* is a word obtained from a B_n reduced word by deleting all the 0's.

Definition. Let $\boldsymbol{\delta}_n$ denote the shifted 'staircase' shape $(2n-2, 2n-4, \dots, 2)$ of size $n(n-1)$. Let its corners be labeled $1, \dots, n-1$ from the bottom row to the top. If T is a standard tableau of shape $\boldsymbol{\delta}_n$, its *promotion sequence* $\hat{p}(T)$ is the sequence $a_1 \dots a_{n(n-1)}$ in which a_i is the label of the corner occupied by the largest entry of $p^{n(n-1)-i}(T)$.

In [**9**] it is shown that $T \to \hat{p}(T)$ defines a bijection from standard tableaux of shape $\boldsymbol{\delta}_n$ to winnowed words for the longest element of B_n and conjectured that initial segments of $\hat{p}(T)$ determine the corresponding initial segments of T. Here we extend these results by proving the conjecture just mentioned and relating the correspondence to D_n.

The first step is to identify both flattened words and winnowed words with words of a third kind. In what follows, flattened words and winnowed words are always for the longest element of D_n or B_n unless mention is made to the contrary. Recall that the longest element of D_n is $w_0^D = \overline{1}\,\overline{2} \dots \overline{n}$ if n is even, or $1\,\overline{2} \dots \overline{n}$ if n is odd.

Definition. A *visiting word* $a_1 \dots a_{n(n-1)}$ is a sequence of symbols $1 \leq a_i < n$ such that

(1) the product $\sigma_{a_1} \cdots \sigma_{a_{n(n-1)}}$ is the identity in the symmetric group S_n, and
(2) for all $k \in \{1, 2, \dots, n\}$, there is a j such that $\sigma_{a_1} \cdots \sigma_{a_j}(1) = k$.

These conditions mean that as the adjacent transpositions $\sigma_{a_1}, \sigma_{a_2}, \dots$ are applied in succession, beginning with the identity permutation $1\ 2 \dots n$, each of the numbers 1 through n visits the leftmost position at some point, and ultimately returns to its original position. Note that $n(n-1)$ is the minimum length for such a sequence, since each number has to switch places twice with every other.

Proposition 3.6. *The sets of visiting words, flattened words, and winnowed words of order n are all the same.*

Proof. Flattening or winnowing a reduced word gives its image under the natural homomorphism from D_n or B_n to S_n in which sign changes are ignored. In D_n and B_n, when the application of $\sigma_{\hat{1}}$ or σ_0 to a (signed) permutation changes the sign of a number, that number must occupy the leftmost position before or after the sign change. From this it is clear that every flattened word and every winnowed word is also a visiting word.

It is also clear that every visiting word is a winnowed word, since to un-winnow it is only necessary for each k to insert a 0 at some point during which k occupies the leftmost position.

The only difficulty is now to see that given a visiting word $\mathbf{a}$, there is always a way of changing some 1's to $\hat{1}$'s to make a reduced word for w_0^D. To $\mathbf{a}$ we associate a graph $G(\mathbf{a})$ with vertex set $\{1, \dots, n\}$ by introducing for each a_j equal to 1 an edge connecting $v_j(1)$ and $v_j(2)$, where $v_j = \sigma_{a_1} \cdots \sigma_{a_j}$. In other words, applying the transpositions σ_{a_i} in succession, each time there is a change in the leftmost position we introduce an edge between the former occupant and its replacement. In general $G(\mathbf{a})$ can have multiple edges, but not loops.

Given a subset of the 1's in $\mathbf{a}$, there is a corresponding subset of the edges in $G(\mathbf{a})$, forming a subgraph H. If we change the 1's in the given subset to $\hat{1}$'s, we get a word describing an element $v \in D_n$ whose unsigned underlying permutation remains the identity. The sign of $v(k)$ is negative if and only if an odd number of edges in H are incident at vertex k, since these edges represent the transpositions $\sigma_{\hat{1}}$ involving k. To un-flatten $\mathbf{a}$, we need $v = w_0^D$; our word will automatically be reduced since its length is $n(n-1)$. Equivalently, we must find a function from the edges of G to $\mathbb{Z}_2$ such that its sum over all incident edges is 1 at every vertex, except possibly vertex 1. It is well-known and easy to prove that a suitable function exists if $G(\mathbf{a})$ is connected.

For each $i \in \{2, \dots, n\}$, let $h(i)$ be the number which i replaces on its first visit to the leftmost position. Note that i and $h(i)$ are linked by an edge of $G(\mathbf{a})$. Moreover $h(i)$ makes its first visit to the leftmost position before i does, showing that the sequence $i, h(i), h(h(i)), \dots$ never repeats and therefore ultimately reaches 1. This proves $G(\mathbf{a})$ is connected. □

From the above proof we can extract something more. The un-flattenings of a given flattened word correspond to solutions of a system of $n-1$ independent linear equations over $\mathbb{Z}_2$ in m variables, where m is the number of edges in $G(\mathbf{a})$. There are 2^{m-n+1} such

solutions. More generally, the same reasoning applies to reduced words for an arbitrary $w \in D_n$, but with $G(\mathbf{a})$ only having vertices for numbers that actually reach the leftmost position. This gives the following result.

Proposition 3.7. *If* $\mathbf{b}$ *is the flattened word of a reduced word for* $w \in D_n$*, then the number of reduced words* $\mathbf{a} \in R(w)$ *which flatten to* $\mathbf{b}$ *is* 2^{m-k+1}*, where* m *is the number of* 1*'s in* $\mathbf{b}$ *and* k *is the number of visitors to the leftmost position, i.e., the number of distinct values taken by* $\sigma_{b_1} \cdots \sigma_{b_j}(1)$ *as* j *varies from* 0 *to* $l(w)$*.*

Note that $m - k + 1$ is the number of *repeat visits* occurring as the transpositions σ_{b_i} are successively applied, *i.e.*, the number of times an application of σ_1 moves a number into the leftmost position which has been there before. In what follows, we denote the number of repeat visits by $r(\mathbf{b})$ and the number $k - 1 = m - r(\mathbf{b})$ of *first visits* by $f(\mathbf{b})$. Abusing notation, we also write $r(\mathbf{a})$ and $f(\mathbf{a})$ for these when $\mathbf{a}$ is a reduced word flattening to $\mathbf{b}$.

To obtain a further corollary to the proof of Proposition 3.6, observe that the subgraph H can be chosen as a subgraph of any given spanning tree of $G(\mathbf{a})$. Indeed, H will then be unique, since it will be given by $k - 1$ independent linear equations in $k - 1$ variables. In particular, the last paragraph of the proof shows that edges of the form $(i, h(i))$ corresponding to first visits form a spanning tree, proving the following.

Proposition 3.8. *If* $\mathbf{b}$ *is a flattened word for* w*, then there is a unique reduced word* $\mathbf{a}$ *for* w *with flattened word* $\mathbf{b}$*, such that all the* $\hat{1}$*'s in* $\mathbf{a}$ *correspond to* 1*'s representing first visits in* $\mathbf{b}$*.*

Now we come to the D_n analog of Proposition 3.3.

Proposition 3.9. (D_n Edelman–Greene correspondence) *The map* $T \mapsto \hat{p}(T)$ *is a bijection from standard tableaux of shape* $\boldsymbol{\delta}_n$ *to flattened words for the longest element of* D_n*. The initial segment* $b_1 \dots b_k$ *of* $\hat{p}(T)$ *determines the initial segment* $T|_k$ *containing entries* 1 *through* k *of* T*. Given a reduced word* $\mathbf{a}$ *with flattened word* $\mathbf{b}$*, denote* $T|_k$ *by* $\Gamma(a_1 \dots a_k)$*. Then the number*

$$e_w^\mu = \sum_{\substack{\mathbf{a} \in R(w) \\ \Gamma(\mathbf{a}) = S}} 2^{-o(\mathbf{a})}, \tag{3.8}$$

where $o(\mathbf{a})$ *denotes the total number of* 1*'s and* $\hat{1}$*'s in* $\mathbf{a}$*, depends only on* w *and on the shape* μ *of* S*. Finally, we have* $P(\mathbf{a}) = P(\Gamma(\mathbf{a}))$ *for the peak sets.*

Proof. The bijection is Theorem 5.16 of [**9**], since we now know that flattened words and winnowed words are the same. The peak set statement follows exactly as in Proposition 3.3 above.

For the assertion about initial segments, we show that whenever $\mathbf{bc}$ and $\mathbf{bc}'$ are flattened words for the longest element, with common initial segment $\mathbf{b}$, then $\mathbf{c}$ and $\mathbf{c}'$ are connected by a chain of S_n Coxeter relations. This given, the proof of Proposition 6.1 in [**9**] applies, with one change. Namely, for the argument involving the Coxeter relation $121 \leftrightarrow 212$ to

go through, when the two flattened words are $\mathbf{b}121$ and $\mathbf{b}212$, their corresponding tableaux must differ only in the largest three entries. But this is shown by the proof of Proposition 5.15 in [**9**].

Now consider two flattened words $\mathbf{bc}$ and $\mathbf{bc}'$. Treating them as winnowed words, note that a winnowed word can be canonically un-winnowed by inserting a 0 at the beginning, and after every 1 that represents a first visit. Since the presence of each 0 is controlled by the initial segment of the word up to that point, the words $\mathbf{bc}$ and $\mathbf{bc}'$ un-winnow to $\mathbf{ad}$ and $\mathbf{ad}'$ for some $\mathbf{a}$, $\mathbf{d}$, and $\mathbf{d}'$ whose winnowed words are $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{c}'$. Then $\mathbf{d}$ and $\mathbf{d}'$ are connected by a chain of B_n Coxeter relations, which after winnowing reduce to S_n Coxeter relations connecting $\mathbf{c}$ to $\mathbf{c}'$.

What remains is to show that the numbers e_w^μ don't depend upon the particular tableau S, only on its shape. Let S and S' be elementary dual equivalent tableaux of shape μ. Let $\mathbf{a}$ be a reduced word for w with $\Gamma(\mathbf{a}) = S$ and let $\mathbf{b}$ be the corresponding flattened word. Note that S is really a function of $\mathbf{b}$ and is the initial segment of any tableau corresponding to an extension of $\mathbf{b}$.

By Lemma 5.2 of [**9**], if we extend S to a tableau T of shape $\boldsymbol{\delta}_n$, and let T' be the corresponding extension of S', then $\hat{p}(T)$ and $\hat{p}(T')$ differ by a certain substitution in the positions corresponding to the segment involved in the elementary dual equivalence $S \approx S'$. The complete list of possible substitutions is given in Table 5 of [**9**].

All but two of these substitutions are special cases of S_n Coxeter relations other than $121 \leftrightarrow 212$. It is easy to see that whenever $\mathbf{b}$ is a flattened word for w and $\mathbf{b}'$ differs from $\mathbf{b}$ by any S_n Coxeter relation besides $121 \leftrightarrow 212$, then $\mathbf{b}'$ is also a flattened word for w and $f(\mathbf{b}') = f(\mathbf{b})$.

The two remaining substitutions are $1121 \leftrightarrow 1212$ and $1211 \leftrightarrow 2121$. For these pairs it is again easy to see that if a flattened word $\mathbf{b}$ for w contains one of the pair, substituting the other yields another flattened word $\mathbf{b}'$ for w. Furthermore, we have $f(\mathbf{b}) = f(\mathbf{b}')$, for the second of the consecutive 1's in 1121 or 1211 never represents a first visit, while the other two bring about visits by the same two numbers as do the two 1's in the substituted 1212 or 2121.

Summarizing, we have bijections between flattened words $\mathbf{b}$ for w with $\Gamma(\mathbf{b}) = S$ and words $\mathbf{b}'$ with $\Gamma(\mathbf{b}') = S'$, and these bijections preserve the number of first visits. Since $2^{r(\mathbf{b})}$ reduced words for w correspond to each flattened word $\mathbf{b}$ we find that the sum

$$\sum_{\substack{\mathbf{b}\in F(w)\\ \Gamma(\mathbf{b})=S}} t^{f(\mathbf{b})} = \sum_{\substack{\mathbf{a}\in R(w)\\ \Gamma(\mathbf{a})=S}} 2^{-r(\mathbf{a})} t^{f(\mathbf{a})} \tag{3.9}$$

is not changed by replacing S with S'. Here $F(w)$ denotes the set of flattened words for w. Since all tableaux of shape μ are connected by chains of elementary dual equivalences, (3.9) depends only on w and μ, and hence so does (3.8), by setting $t = 1/2$. □

Now we have the D_n analog of (3.5).

Definition. Let w be an element of D_n. The D_n *Stanley function* $E_w(X)$ is defined by

$$E_w(X) = \sum_\mu e_w^\mu Q_\mu(X). \tag{3.10}$$

Just as for B_n, we immediately obtain an identity from Propositions 3.2 and 3.9, and the corresponding corollary, with the same proof as Corollary 3.5.

Proposition 3.10.

$$\begin{aligned} E_w(X) &= \sum_{\mathbf{a}\in R(w)} 2^{-o(\mathbf{a})}\Theta^{l(w)}_{P(\mathbf{a})}(X) \\ &= \sum_{\mathbf{a}\in R(w)} \sum_{\substack{(i_1\le\cdots\le i_l)\\ \in A(P(\mathbf{a}))}} 2^{|\mathbf{i}|-o(\mathbf{a})} x_{i_1}x_{i_2}\cdots x_{i_l}. \end{aligned} \tag{3.11}$$

Corollary 3.11. *For all w, $E_w(X) = E_{w^{-1}}(X)$.*

Although the coefficients e_w^μ need not be integers, it is nevertheless true that $E_w(X)$ is an integral linear combination of P-functions, as we show next.

For this purpose we must extract a concept which is implicit in the proof of Proposition 3.9. We define flattened words $\mathbf{b}$ and $\mathbf{b}'$ to be *dual equivalent* if they are connected by a chain of substitutions from Table 5 of [**9**]. The proof of Proposition 3.9 shows that $\mathbf{b}$ and $\mathbf{b}'$ are then flattened words for the same elements w, and that the map Γ is a bijection from each dual equivalence class to the set of all standard tableaux of some shape μ. Moreover, $f(\mathbf{b})$ is constant on dual equivalence classes.

Proposition 3.12. *The Stanley functions $E_w(X)$ are integral linear combinations of Schur P-functions.*

Proof. This amounts to saying that $2^{l(\mu)}e_w^\mu$ is an integer. Since e_w^μ is given by (3.9) with $t = 1/2$, and (3.9) is a polynomial with integer coefficients, it suffices to show that $l(\mu) \ge f(\mathbf{b})$ for every flattened word $\mathbf{b}$ such that $\Gamma(\mathbf{b})$ has shape μ.

Since both μ and $f(\mathbf{b})$ are constant on dual equivalence classes, we can assume that $\Gamma(\mathbf{b})$ is the tableau T_0 formed by numbering the cells of μ from left to right, one row at a time. The peaks of T_0 occur at the end of each row except the last, so $|P(T_0)| = |P(\mathbf{b})| = l(\mu) - 1$. For any flattened word, we have $f(\mathbf{b}) \le |P(\mathbf{b})| + 1$, since each first visit is represented by a 1 in $\mathbf{b}$, no two of these 1's can be consecutive, and between every two non-consecutive 1's there is at least one peak. This shows $f(\mathbf{b}) \le l(\mu)$, as required. □

To close, we evaluate E_w and F_w for some special values of w.

Proposition 3.13. *Let $\mu = (\mu_1 > \cdots > \mu_l)$, where μ_l is taken to be zero if necessary to make the number of parts even. Let $\nu_i = \mu_i + 1$ and let $w = \overline{\nu_1}\ \overline{\nu_2}\ldots\overline{\nu_l}12\ldots$. Then $E_w(X) = P_\mu(X)$.*

Proof. Our method is to give an explicit description of the reduced words for w and compute E_w directly. In order to do this, we introduce a new bijection ϕ, different from Γ, from reduced words for w to standard tableaux of shape μ.

For any element v of D_n the *inversions* of v are (1) pairs $i < j$ for which $v(i) > v(j)$; (2) pairs $i < j$ for which the larger in absolute value of $v(i)$ and $v(j)$ is negative. (A pair can count twice, once in each category). The length $l(v)$ is the number of inversions. In particular, we have $l(w) = |\mu|$ since there are no inversions of type (1) and each i, $1 \le i \le l$, is involved in μ_i inversions of type (2). Let $m = |\mu| = l(w)$.

We now claim that $\mathbf{a} = a_1 a_2 \dots a_m$ is a reduced word for w if and only if at every stage j, applying σ_{a_j} to the signed permutation $\sigma_{a_1} \cdots \sigma_{a_{j-1}}$ does one of two things:

(1) moves one of the numbers ν_k which is still positive at this stage to the left across a number which is not a positive ν_i, or

(2) if the smallest two currently positive ν_i's occupy positions 1 and 2, applies $\sigma_{\hat{1}}$ to exchange them and make them negative.

To justify the claim, we note first that such a sequence of operations clearly realizes w after m steps, hence $\mathbf{a}$ is a reduced word for w. To see that every reduced word for w has this form, it is only necessary to check that the form is preserved when $\mathbf{a}$ is modified by any D_n Coxeter relation. For this, note that Coxeter relations of the form $aba \leftrightarrow bab$ with a, b adjacent never apply, nor does $1\hat{1} \leftrightarrow \hat{1}1$. For all others, of the form $ac \leftrightarrow ca$ with a, c nonadjacent, the verification is trivial.

Now, given a reduced word $\mathbf{a}$ for w, let $v_j = \sigma_{a_1} \cdots \sigma_{a_j}$. Let k_j be the number of ν_i's which appear with positive sign and not in position 1 in the signed permutation v_j, and let λ_j be the partition whose parts are one less than the positions of these ν_i's, a partition with k_j distinct parts. Observe that in passing from v_j to v_{j+1} by move (1) or (2) above, exactly one part of λ_j is reduced by 1 to give λ_{j+1}, and the available choices for a move correspond one-to-one with the corners of the Ferrers diagram of λ_j . Also observe that $\lambda_0 = \mu$. Therefore the sequence of shapes $\emptyset = \lambda_m \subset \lambda_{m-1} \subset \cdots \subset \lambda_0 = \mu$ describes the initial segments of a unique standard tableau $\phi(\mathbf{a})$ of shape μ, every standard tableau occurs, and the tableau contains sufficient information to reconstruct the sequence of moves and thus $\mathbf{a}$. This shows ϕ is a bijection from reduced words for w to standard tableaux of shape μ.

Note that $m - j$ is a descent of $\phi(\mathbf{a})$ if and only if the move made at stage j occurs to the left of the move made at stage $j+1$. This shows that the descent set $D(\phi(\mathbf{a}))$ is the same as that of the reversed reduced word $\mathbf{a}^r = a_m a_{m-1} \dots a_1$. Hence their peak sets are also equal. Note also that each reduced word contains a total of l 1's and $\hat{1}$'s, all representing first visits, so there is one reduced word per flattened word, or in other words, the flattenings of the reduced words are all distinct.

Formula (3.11) for $E_{w^{-1}}$ thus reduces to

$$2^{-l} \sum_{\mathrm{sh}\, T = \mu} \Theta^{|\mu|}_{P(T)}(X), \tag{3.12}$$

which is $P_\mu(X)$ by Proposition 3.2. Since $E_w = E_{w^{-1}}$ by Corollary 3.11, the proof is com-

plete. □

Proposition 3.14. *Let* $\mu = (\mu_1 > \cdots > \mu_l)$ *and let* $w = \overline{\mu_1}\,\overline{\mu_2}\cdots\overline{\mu_l}12\ldots$. *Then* $F_w(X) = Q_\mu(X)$.

Proof. Since the argument here is virtually identical to that used for preceding proposition, we only give a sketch.

Again we have $l(w) = |\mu|$ by straightforward considerations. (Inversions for B_n are the same as those for D_n, plus one for every negative $v(i)$.)

In this case the allowable "moves" associated with a reduced word are:

(1) move a currently positive μ_k left across anything except a positive μ_i, or

(2) if a positive μ_i occupies position 1, apply σ_0 to change its sign.

The tableau $\phi(\mathbf{a})$ is formed from a sequence of shapes λ_j exactly as before, except now the parts of λ_j are the positions of all the positive μ_i's (including in position 1, and without subtracting one). This ϕ is a bijection exactly as before, and again we have $P(\phi(\mathbf{a})) = P(\mathbf{a}^r)$. Hence using formula (3.6) for $F_{w^{-1}}$, Proposition 3.2, and Corollary 3.5, we find $F_w = Q_\mu$ as asserted. □

For our remaining special case computations we require some facts about the unshifted Edelman–Greene correspondence.

Definition. Let $\boldsymbol{\alpha}_n$ denote the straight (*i.e.*, not shifted) staircase shape $(n-1, n-2, \ldots, 1)$, of size $\binom{n}{2}$. Let its corners be labeled $1, 2, \ldots, n-1$ from bottom to top. If T is a standard tableau of shape $\boldsymbol{\alpha}_n$, its *promotion sequence* $\hat{p}(T)$ is the sequence $a_1 \ldots a_{\binom{n}{2}}$ in which a_i is the label of the corner occupied by the largest entry of $p^{\binom{n}{2}-i}(T)$.

Proposition 3.15. (Edelman–Greene correspondence) *The map* $T \to \hat{p}(T)$ *is a bijection from standard tableaux of shape* $\boldsymbol{\alpha}_n$ *to reduced words for the longest element of* S_n. *The initial segment* $a_1 \ldots a_k$ *of* $\hat{p}(T)$ *determines the initial segment* $T|_k$ *containing entries* 1 *through* k *of* T. *Denoting* $T|_k$ *by* $\Gamma(a_1 \ldots a_k)$, *the number*

$$g_v^\lambda = |\{\mathbf{a} \in R(v) \mid \Gamma(\mathbf{a}) = S\}| \tag{3.13}$$

depends only on v *and on the shape* λ *of* S. *We have* $D(\mathbf{a}) = D(\Gamma(\mathbf{a}))$ *for the descent sets.*

Definition. Let v be an element of S_n. The *Stanley function* $G_v(X)$ is defined by

$$G_v(X) = \sum_\lambda g_v^\lambda s_\lambda(X), \tag{3.14}$$

where s_λ denotes the usual Schur S-function.

Proposition 3.15 is proved in [**9**], where it is also shown that the above definition of S_n Stanley functions agrees with the original definition in [**17**]. (In [**17**], and also in Chapter 7 of [**13**], where G_v is shown to be a 'stable' Schubert polynomial of type A, G_v and g_v^λ are denoted F_v and $\alpha(\lambda, v)$.)

Now we can express the functions $F_{w_0^B v}$ and $E_{w_0^D v}$, for $v \in S_n$, in terms of the quantities just defined.

Proposition 3.16. *Let w_0^B, w_0^D, and v_0 be the longest elements of B_n, D_n, and S_n, respectively. Let δ_k denote the partition $(k, k-1, \ldots, 1)$. Then we have for every $v \in S_n$*

$$F_{w_0^B v} = \sum_\lambda g^\lambda_{v_0 v} Q_{\delta_n + \lambda}, \tag{3.15}$$

$$E_{w_0^D v} = \sum_\lambda g^\lambda_{v_0 v} P_{\delta_{n-1} + \lambda}. \tag{3.16}$$

Equivalently, $F_{w_0^B v}$ and $E_{w_0^D v}$ are the images of $G_{v_0 v}$ under linear transformations sending Schur functions s_λ to $Q_{\delta_n+\lambda}$ and $P_{\delta_{n-1}+\lambda}$, respectively.

Proof. Fix a reduced word $\mathbf{c}$ for v^{-1}. Then the reduced words for $v_0 v$ are exactly the initial parts $\mathbf{a}$ of those reduced words $\mathbf{ac}$ for v_0 which end in $\mathbf{c}$. Similar statements apply with w_0^B and w_0^D in place of v_0.

From this observation and Proposition 3.15 it follows that $g^\lambda_{v_0 v}$ is equal to the number of tableaux S of skew shape $\boldsymbol{\alpha}_n/\lambda$ for which $\hat{p}(S) = \mathbf{c}$. Similarly, $f^\mu_{w_0^B v}$ is the number of tableaux T of shape $\boldsymbol{\beta}_n/\mu$ for which $\hat{p}(T) = \mathbf{c}$. But there are no 0's in $\mathbf{c}$, and therefore $f^\mu_{w_0^B v}$ is non-zero only if the shape μ contains the corner with label 0, that is, if $\mu = \delta_n + \lambda$ for some λ. In this case, the rules for computing $\hat{p}(S)$ and $\hat{p}(T)$ are identical, showing that $f^{\delta_n+\lambda}_{w_0^B v} = g^\lambda_{v_0 v}$. This proves (3.15).

For (3.16), we need $2^{n-1} e^{\delta_{n-1}+\lambda}_{w_0^D v} = g^\lambda_{v_0 v}$ and $e^\mu_{w_0^D v} = 0$ if μ is not of the form $\delta_{n-1} + \lambda$. Since $(w_0^D v)^{-1}(k)$ is negative for all $k \in \{2, \ldots, n\}$, we have $f(\mathbf{a}) = n-1$ for all $\mathbf{a} \in R(w_0^D v)$. In the proof of Proposition 3.12 we showed that $f(\mathbf{b}) \le l(\mu)$ whenever $\Gamma(\mathbf{b})$ has shape μ. This shows $e^\mu_{w_0^D v}$ is non-zero only for μ of the form $\delta_{n-1} + \lambda$. Moreover, using the left-hand side of (3.9) with $t = 1/2$ to evaluate $e^\mu_{w_0^D v}$, we find that $2^{n-1} e^\mu_{w_0^D v}$ is the number of flattened words $\mathbf{b}$ for $w_0^D v$ with $\Gamma(\mathbf{b}) = S$, for any given tableau S of shape μ.

If $\mathbf{b}$ is a flattened word for $w_0^D v$, then $\mathbf{bc}$ is clearly a flattened word for w_0^D. Every element of S_n has a reduced word containing at most one 1, so we may choose $\mathbf{c}$ with this property. For such $\mathbf{c}$, we claim the converse holds: if $\mathbf{bc}$ is a flattened word for w_0^D then $\mathbf{b}$ is a flattened word for $w_0^D v$. This amounts to saying that $\mathbf{bc}$ can be un-flattened without changing any 1's in $\mathbf{c}$ to $\hat{1}$'s. If there are no 1's in $\mathbf{c}$, this is trivial. If there is a single 1, then it is the last 1 in the visiting word $\mathbf{bc}$, corresponding to the transposition moving 1 into the leftmost position for the last time. As such, it represents a repeat visit, so Proposition 3.8 shows we can un-flatten $\mathbf{bc}$ without changing it to a $\hat{1}$.

In view of the claim just proven, Proposition 3.9 shows that $2^{n-1} e^\mu_{w_0^D v}$ is equal to the number of tableaux T of shape $\boldsymbol{\delta}_n/\mu$ for which $\hat{p}(T) = \mathbf{c}$. Exactly as in the argument above for B_n, this is the same as $g^\lambda_{v_0 v}$ for $\mu = \delta_{n-1} + \lambda$. $\square$

We give one final special case evaluation for its inherent interest, even though we will not need it later.

Proposition 3.17. *Let ϕ be the homomorphism from the ring of symmetric functions onto the subring generated by odd power sums defined by*

$$\phi(p_k) = \begin{cases} 2p_k & \text{for } k \text{ odd,} \\ 0 & \text{for } k \text{ even.} \end{cases} \tag{3.17}$$

Then for $v \in S_n$, we have

$$F_v = \phi(G_v), \tag{3.18}$$

and if in addition $v(1) = 1$,

$$E_v = \phi(G_v). \tag{3.19}$$

Proof. If $\mathbf{a}$ is a reduced word for $v \in S_n$, let us denote the corresponding tableaux $\Gamma(\mathbf{a})$ under the A_n, B_n, and D_n Edelman–Greene correspondences by $\Gamma_A(\mathbf{a})$, $\Gamma_B(\mathbf{a})$, and $\Gamma_D(\mathbf{a})$. It is easy to show that $\Gamma_B(\mathbf{a})$ and $\Gamma_D(\mathbf{a})$ are both identical to the tableau obtained by bringing $\Gamma_A(\mathbf{a})$ to normal shifted shape via shifted *jeu-de-taquin*. Hence for S of shape μ, $f_v^\mu = |\{\mathbf{a} \in R(v) \mid \Gamma_B(\mathbf{a}) = S\}| = \sum_\lambda k_\lambda^\mu g_v^\lambda$, where k_λ^μ is the number of standard tableaux of straight shape λ carried by shifted *jeu-de-taquin* to any given tableau of shifted shape μ.

In [**20**] it is shown that $\phi(s_\lambda) = \sum_\mu k_\lambda^\mu Q_\mu$. Equation (3.18) follows immediately. Equation (3.19) follows because when $v(1) = 1$, there are no 1's in any reduced word for v, and therefore $e_v^\mu = f_v^\mu$. □

4. Proofs of the Main Theorems

In this section we prove Theorems 1 through 4. Our central results are contained in Theorems 3 and 4. Below we have split each theorem into three separate statements, labeled A, B, C. Theorems 3A and 4A are the promised formulas for Schubert polynomials. Parts B and C are the additional results that the Schubert polynomials form a $\mathbb{Z}$-basis for the relevant ring and that they reduce to Schur P- and Q-functions in the special cases corresponding to Schubert classes for isotropic Grassmannians. We conclude the section with some auxiliary results useful for computing Schubert polynomials of type C and D, as well as tables for the groups B_3 and D_3.

Theorem 1. *Solutions of the defining equations for each type of Schubert polynomial exist and are unique.*

Proof. The existence follows from Theorems 2, 3A, and 4A. To show uniqueness, let $\{S_w\}$ be a family of polynomials satisfying the defining recurrence relations (2.2), (2.3), or (2.4),

together with the constant term conditions. Suppose $\{S'_w\}$ is another solution. For each i,

$$\partial_i\big(S_w - S'_w\big) = \begin{cases} S_{w\sigma_i} - S'_{w\sigma_i} & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w). \end{cases} \tag{4.1}$$

By induction on the length of w, we may assume $S_{w\sigma_i} - S'_{w\sigma_i} = 0$. Then $\partial_i\big(S_w - S'_w\big) = 0$ for each i, so $S_w - S'_w$ is invariant for the relevant group S_∞, B_∞ or D_∞. The only S_∞ invariants in $\mathbb{Q}[z_1, z_2, \dots]$ are constants, as are the B_∞ or D_∞ invariants in $\mathbb{Q}[z_1, z_2, \dots ; p_1, p_3, \dots]$, because the even power sums are missing. Hence $S_w - S'_w$ is constant, so $S_w = S'_w$ by the constant term conditions. □

Theorem 2. *The Schubert polynomials of type A are the same as those defined by Lascoux–Schützenberger* [**12**]. *Namely, for each* $w \in S_n$,

$$\mathfrak{S}_w = \partial_{a_1}\partial_{a_2}\cdots\partial_{a_l}(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}), \tag{4.2}$$

where $a_1a_2\dots a_l \in R(w_0w^{-1})$ *and* $w_0 = (n, n-1, \dots, 1)$ *is the permutation of longest length in* S_n.

Proof. In [**12**] it is shown that $\mathfrak{S}_w$ is well-defined by (4.2), independent of n, and satisfies the defining recurrence (2.2). For complete details, see Chapter 4 of [**13**]. □

Theorems 3A and 4A involve 'admissible monomial' forms of formulas (2.5) and (2.8), derived from formulas (3.6) and (3.11) and the admissible monomial formula for type A Schubert polynomials, (4.5) below. In order to distinguish between two notions of admissibility we will make the following conventions. If $\mathbf{a} = a_1a_2\dots a_l$ is a reduced word for an element $w \in B_\infty$ or D_∞, we let $\mathcal{A}_x(\mathbf{a})$ denote the set of admissible monomials $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m} = x_{i_1}x_{i_2}\cdots x_{i_l}$, such that $(i_1 \le i_2 \le \cdots \le i_l) \in A(P(\mathbf{a}^r))$, where $\mathbf{a}^r$ is the reversed sequence $a_l\cdots a_2a_1$. Equivalently, $\mathcal{A}_x(\mathbf{a})$ consists of monomials $x^\alpha = x_{j_1}\cdots x_{j_l}$ for which

(1) $j_1 \ge j_2 \ge \cdots \ge j_l$
(2) $j_{k-1} = j_k = j_{k+1}$ implies $k \notin P(\mathbf{a})$.

By Corollary 3.5 we have for $w \in B_n$,

$$F_w(X) = \sum_{\substack{\mathbf{a}\in R(w) \\ x^\alpha \in \mathcal{A}_x(\mathbf{a})}} 2^{i(\alpha)}x^\alpha, \tag{4.3}$$

where $i(\alpha)$ is the number of distinct variables with non-zero exponent in x^α. By Corollary 3.11, we have

$$E_w(X) = \sum_{\substack{\mathbf{a}\in R(w) \\ x^\alpha \in \mathcal{A}_x(\mathbf{a})}} 2^{i(\alpha)-o(\mathbf{a})}x^\alpha \tag{4.4}$$

for each $w \in D_n$.

If $\mathbf{a} = a_1a_2\dots a_l$ is a reduced word for $w \in S_n$, we let $\mathcal{A}_z(\mathbf{a})$ denote the set of monomials $z^\alpha = z_{j_1}\cdots z_{j_l}$ satisfying the following admissibility constraints:

(1) $j_1 \leq j_2 \leq \cdots \leq j_l$
(2) $j_i = j_{i+1}$ implies $a_i > a_{i+1}$
(3) $j_i \leq a_i$ for all i.

Proposition 4.1. [**3**] *For all $w \in S_\infty$,*

$$\mathfrak{S}_w(z_1, z_2, \dots) = \sum_{\substack{\mathbf{a} \in R(w) \\ z^\alpha \in \mathcal{A}_z(\mathbf{a})}} z^\alpha. \tag{4.5}$$

Theorem 3A. *The Schubert polynomials $\mathfrak{C}_w$ are given by the two equivalent formulas:*

$$\mathfrak{C}_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v \in S_\infty}} F_u(X)\mathfrak{S}_v(Z) \tag{4.6}$$

$$= \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v \in S_\infty}} \sum_{\substack{\mathbf{a} \in R(u) \\ x^\alpha \in \mathcal{A}_x(\mathbf{a})}} \sum_{\substack{\mathbf{b} \in R(v) \\ z^\beta \in \mathcal{A}_z(\mathbf{b})}} 2^{i(\alpha)} x^\alpha z^\beta. \tag{4.7}$$

Theorem 4A. *The Schubert polynomials $\mathfrak{D}_w$ are given by the two equivalent formulas:*

$$\mathfrak{D}_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v \in S_\infty}} E_u(X)\mathfrak{S}_v(Z) \tag{4.8}$$

$$= \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v \in S_\infty}} \sum_{\substack{\mathbf{a} \in R(u) \\ x^\alpha \in \mathcal{A}_x(\mathbf{a})}} \sum_{\substack{\mathbf{b} \in R(v) \\ z_\beta \in \mathcal{A}_z(\mathbf{b})}} 2^{i(\alpha)-o(\mathbf{a})} x^\alpha z^\beta. \tag{4.9}$$

We prove several lemmas before proving Theorems 3A and 4A.

Lemma 4.2. *For any f and g, and any i, we have*

$$\partial_i(fg) = (\partial_i f)(\sigma_i g) + f\ \partial_i g. \tag{4.10}$$

Proof. Expand both sides and observe they are equal. □

Lemma 4.3. *Let W denote any of the groups S_∞, B_∞, or D_∞. Let $G_u(Z)$ be arbitrary symmetric functions indexed by elements $u \in W$ and define*

$$H_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v \in S_\infty}} G_u(Z)\mathfrak{S}_v(Z). \tag{4.11}$$

Then for all $i > 0$ and $w \in W$,

$$\partial_i H_w = \begin{cases} H_{w\sigma_i} & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w). \end{cases} \tag{4.12}$$

Proof. For $i > 0$, the operator ∂_i commutes with multiplication by the symmetric function $G_u(Z)$. Hence,

$$\partial_i H_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v\in S_\infty}} G_u(Z)\partial_i \mathfrak{S}_v(Z) \tag{4.13}$$

$$= \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v\in S_\infty \\ l(v\sigma_i)<l(v)}} G_u(Z)\mathfrak{S}_{v\sigma_i}(Z) \tag{4.14}$$

by the defining recurrence for the Schubert polynomials $\mathfrak{S}_w$.

If $l(w\sigma_i) > l(w)$ then the conditions $uv = w$, $l(u) + l(v) = l(w)$, $v \in S_\infty$ and $l(v\sigma_i) < l(v)$ are never satisfied, so (4.14) is equal to zero.

On the other hand, if $l(w\sigma_i) < l(w)$ then the map $(u, v) \mapsto (u, v\sigma_i)$ is a bijection from all (u, v) such that $uv = w$, $l(u)+l(v) = l(w)$, $v \in S_\infty$, and $l(v\sigma_i) < l(v)$ to all (u', v') such that $u'v' = w\sigma_i$, $l(u') + l(v') = l(w\sigma_i)$, $v' \in S_\infty$. Therefore (4.14) is $H_{w\sigma_i}$. □

From here out, symmetric functions in X depend only on odd power sums and really represent symmetric functions in Z via the relation $p_k(X) = -p_k(Z)/2$.

Lemma 4.4. *For any symmetric function $G(X)$ belonging to the ring generated by odd power sums, we have*

$$\sigma_0 G(X) = G(z_1 + X) \tag{4.15}$$

$$\partial_0 G(X) = \frac{G(X) - G(z_1 + X)}{-2z_1}, \tag{4.16}$$

where $G(z_1 + X) = G(z_1, x_1, x_2, \dots)$.

Proof. Because σ_0 is a ring homomorphism, it suffices to verify (4.15) for $G(X) = p_k(X)$, an odd power sum.

$$\begin{aligned} \sigma_0 p_k(X) &= -\frac{1}{2}\sigma_0 p_k(Z) \\ &= -\frac{1}{2}p_k(-z_1, z_2, \dots) \\ &= z_1^k - \frac{1}{2}p_k(z_1, z_2, \dots) \\ &= z_1^k + p_k(X) \\ &= p_k(z_1 + X). \end{aligned} \tag{4.17}$$

Equation (4.16) follows from (4.15). □

We will not be using the next corollary for the proofs that follow. However, it is useful for computing tables of Schubert polynomials.

Corollary 4.5. *The action of ∂_0 on $Q_\mu(X)$ is given by*

$$\partial_0 Q_\mu(X) = \sum_{0<k\leq\mu_1} Q_{\mu/(k)}(X) z_1^{k-1}. \tag{4.18}$$

Proof. Q_μ belongs to the ring generated by odd power sums, so Lemma 4.4 applies. We have $Q_\mu(z_1 + X) = \sum_{\lambda\subseteq\mu} Q_\lambda(z_1) Q_{\mu/\lambda}(X)$, where $Q_{\mu/\lambda}$ for a skew shifted shape is given by the obvious extension of Proposition 3.2. The factor $Q_\lambda(z_1)$ is equal to zero unless $\lambda = (k)$ is a one row shape, in which case it is equal to 1 if $k = 0$ and $2z_1^k$ if $k > 0$. Therefore,

$$\begin{aligned} \partial_0 Q_\mu(X) &= \frac{Q_\mu(X) - Q_\mu(z_1 + X)}{-2z_1} \\ &= \frac{Q_\mu(X) - Q_\mu(X) - \sum_{0<k\leq\mu_1} Q_{\mu/(k)} 2z_1^k}{-2z_1}, \end{aligned} \tag{4.19}$$

which simplifies to (4.18). □

Definition. A reduced word $\mathbf{a} = a_1 a_2 \dots a_l$ and monomial $x^\alpha \in \mathcal{A}_x(\mathbf{a})$ will be referred to as a *reduced word admissible monomial pair*, and denoted by $\binom{\mathbf{a}}{x^\alpha}$. Similarly, $\left[\begin{smallmatrix}\mathbf{c}\\ z^\gamma\end{smallmatrix}\right]$ will denote a reduced word admissible monomial pair if $z^\gamma \in \mathcal{A}_z(\mathbf{c})$.

The notation is merely a bookkeeping device to exhibit the reduced word associated with a particular term. By our convention, when these symbols appear in a polynomial the value of $\binom{\mathbf{a}}{x^\alpha}$ is x^α. We multiply the symbols by concatenating the reduced words and multiplying the monomials. Note the use of the notation $\left[\begin{smallmatrix}\mathbf{c}\\ z^\gamma\end{smallmatrix}\right]$ implies that 0 and $\hat{1}$ do not appear in the reduced word $\mathbf{c}$, by the definition of $\mathcal{A}_z$. With this notation, (4.7) becomes

$$\mathfrak{C}_w = \sum_{\mathbf{ab}\in R(w)} \sum_{\substack{x^\alpha\in\mathcal{A}_x(\mathbf{a})\\ z^\beta\in\mathcal{A}_z(\mathbf{b})}} 2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \begin{bmatrix}\mathbf{b}\\ z^\beta\end{bmatrix} \tag{4.20}$$

$$= \sum_{\mathbf{ab}\in R(w)}{\sum}^{\sim} 2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \begin{bmatrix}\mathbf{b}\\ z^\beta\end{bmatrix}. \tag{4.21}$$

Here the symbol $\widetilde{\sum}$ indicates that the sum ranges over all possible admissible monomials for each $\mathbf{a}$ and $\mathbf{b}$.

Lemma 4.6. *For any reduced word admissible monomial pair $\left[\begin{smallmatrix}\mathbf{c}\\ z^\gamma\end{smallmatrix}\right]$, where $z^\gamma = z_1^{\gamma_1} z_2^{\gamma_2} \cdots$, we have*

$$\partial_0 \begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix} = \begin{cases} -\frac{1}{z_1}\left[\begin{smallmatrix}\mathbf{c}\\ z^\gamma\end{smallmatrix}\right] & \gamma_1 \text{ odd} \\ 0 & \gamma_1 \text{ even.} \end{cases} \tag{4.22}$$

Proof. Equation (4.22) follows immediately from the definition $\partial_0 f = (f - \sigma_0 f)/(-2z_1)$. □

Recall that a sequence $b_1 > b_2 > \cdots > b_j < \cdots < b_k$ having no peak is said to be a *vee*.

Lemma 4.7. *For all $u \in B_\infty$,*

$$\partial_0 F_u(X) = \frac{1}{z_1} \sum_{\substack{\mathbf{ab}\in R(u)\\ k>0}}^{\sim} 2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \binom{\mathbf{b}}{z_1^k}, \tag{4.23}$$

where the notation signifies that $\mathbf{a}$ *and* $\mathbf{b}$ *range over reduced word admissible monomial pairs such that* $\mathbf{ab} \in R(u)$ *and* $\mathbf{b}$ *is a vee of length* $k > 0$.

Proof. By Lemma 4.4,

$$\partial_0 F_u(X) = \frac{F_u(X) - F_u(z_1 + X)}{-2z_1} \tag{4.24}$$

$$= \frac{1}{-2z_1}\Big[\sum_{\mathbf{c}\in R(u)}^{\sim} 2^{i(\gamma)} \binom{\mathbf{c}}{x^\gamma} - \sum_{\mathbf{ab}\in R(u)}^{\sim} 2^{i(\alpha)+\chi_k} \binom{\mathbf{a}}{x^\alpha} \binom{\mathbf{b}}{z_1^k} \Big], \tag{4.25}$$

where χ_k is 1 if $k > 0$ and 0 otherwise. In the second sum $\mathbf{a}$ and $\mathbf{b}$ range over all pairs such that $\mathbf{ab} \in R(u)$ and $\mathbf{b}$ is a vee; k is the length of $\mathbf{b}$. Note that the requirement that $\mathbf{b}$ is a vee is implicit in the use of the symbol $\binom{\mathbf{b}}{z_1^k}$. The terms in the first sum are just the terms with $k = 0$ in the second sum. Hence

$$\partial_0 F_u(X) = \frac{1}{2z_1} \sum_{\substack{\mathbf{ab}\in R(u)\\ k>0}}^{\sim} 2^{i(\alpha)+1} \binom{\mathbf{a}}{x^\alpha} \binom{\mathbf{b}}{z_1^k}, \tag{4.26}$$

which is the same as (4.23). □

Lemma 4.8. *Let $\tilde{\mathfrak{C}}_w$ denote the polynomial defined by* (4.6) *and* (4.7). *For all $w \in B_\infty$,*

$$\partial_0 \tilde{\mathfrak{C}}_w = \frac{1}{z_1} \sum_{\substack{\mathbf{abc}\in R(w)\\ b_k=0}}^{\sim} 2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \begin{bmatrix} b_1 b_2 \dots b_{k-1} \\ z_1^{k-1} \end{bmatrix} \binom{b_k}{z_1} \begin{bmatrix} \mathbf{c} \\ z_2^{\gamma_2} z_3^{\gamma_3} \dots \end{bmatrix}, \tag{4.27}$$

where the notation implies the restriction $b_1 > b_2 > \cdots > b_k = 0$ and $\mathbf{b} = b_1 b_2 \dots b_k$ has length $k > 0$.

Proof. From the definition, we have

$$\partial_0 \tilde{\mathfrak{C}}_w = \sum_{\substack{uv=w\\ l(u)+l(v)=l(w)\\ v\in S_\infty}} \sum_{\mathbf{c}\in R(v)}^{\sim} \partial_0 \Big(F_u(X) \begin{bmatrix} \mathbf{c} \\ z^\gamma \end{bmatrix} \Big). \tag{4.28}$$

By Lemma 4.2, we can expand (4.28) as the sum of two polynomials. The first term of (4.10) yields

(4.29)
$$\sum_{\substack{uv=w\\ l(u)+l(v)=l(w)\\ v\in S_\infty}} \sum_{\mathbf{c}\in R(v)}^{\sim} \Big(\partial_0 F_u(X)\Big)\Big(\sigma_0 \begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix}\Big)$$

$$= \frac{1}{z_1} \sum_{\substack{\mathbf{abc}\in R(w)\\ k>0}}^{\sim} (-1)^{\gamma_1} 2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \binom{b_1 b_2 \dots b_k}{z_1^k} \begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix} \tag{4.30}$$

$$= \frac{1}{z_1} \sum_{\substack{\mathbf{abc}'\in R(w)\\ k>0}}^{\sim} (-1)^{\gamma_1} 2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \binom{b_1 b_2 \dots b_k}{z_1^k} \begin{bmatrix}b_{k+1}\dots b_m\\ z_1^{\gamma_1}\end{bmatrix} \begin{bmatrix}\mathbf{c}'\\ z_2^{\gamma_2} z_3^{\gamma_3}\dots\end{bmatrix}. \tag{4.31}$$

The second term of (4.10) yields

(4.32)
$$\sum_{\substack{uv=w\\ l(u)+l(v)=l(w)\\ v\in S_\infty}} \sum_{\mathbf{c}\in R(v)}^{\sim} F_u(X)\Big(\partial_0 \begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix}\Big) = \frac{1}{z_1} \sum_{\substack{\mathbf{ac}\in R(w)\\ \gamma_1 \text{ odd}}}^{\sim} -2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix}$$

$$= \frac{1}{z_1} \sum_{\substack{\mathbf{abc}'\in R(w)\\ m \text{ odd}}}^{\sim} -2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \begin{bmatrix}\mathbf{b}\\ z_1^m\end{bmatrix} \begin{bmatrix}\mathbf{c}'\\ z_2^{\gamma_2} z_3^{\gamma_3}\dots\end{bmatrix}. \tag{4.33}$$

Next we examine the coefficient C_A of the general term

$$A = \frac{1}{z_1} 2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \left\langle \begin{matrix} b_1 \dots b_m \\ z_1^m \end{matrix} \right\rangle \begin{bmatrix}\mathbf{c}\\ z_2^{\gamma_2} z_3^{\gamma_3}\dots\end{bmatrix} \tag{4.34}$$

in the sum of (4.31) and (4.33). Here the bracket $\left\langle\right\rangle$ denotes the entire factor involving z_1, for which $\mathbf{b}$ is, in general, a vee followed by a decreasing sequence. From (4.31) there is a contribution of $(-1)^{\gamma_1} = (-1)^{m-k}$ to C_A for every k such that

$$\left\langle \begin{matrix} b_1 \dots b_m \\ z_1^m \end{matrix} \right\rangle = \binom{b_1 \dots b_k}{z_1^k} \begin{bmatrix} b_{k+1} \dots b_m \\ z_1^{m-k} \end{bmatrix}, \tag{4.35}$$

i.e., such that $b_1 \dots b_k$ is a vee, $b_{k+1} > \dots > b_m$ and $b_m \neq 0$ unless $k = m$. From (4.33) there is a contribution of -1 provided $b_1 > \dots > b_m$, m is odd, and $b_m \neq 0$.

We need to verify that $C_A = 0$ unless $b_1 > \dots > b_m = 0$, and then $C_A = 1$.

Case 1: $b_m \neq 0$

First assume there is an index i such that $b_i < b_{i+1}$, and choose i to be as large as possible. Then (4.31) contributes two terms, for $k = i$ and $k = i+1$, which cancel, while (4.33) contributes nothing.

Otherwise, assume $b_1 > b_2 > \cdots > b_m > 0$. For each $1 \le k \le m$, there is a contribution of $(-1)^{m-k}$ from (4.31). If m is even, then $C_A = \sum_{k=1}^{m}(-1)^{m-k} = 0$. If m is odd there is also a contribution from (4.33) so $C_A = -1 + \sum_{k=1}^{m}(-1)^{m-k} = 0$. Therefore, every term A with $b_m \neq 0$ has $C_A = 0$.

Case 2: $b_m = 0$

In this case, the only contribution is from (4.35) with $k = m$, *i.e.* $\left\langle {b_1 \dots b_m \atop z_1^m} \right\rangle = \binom{b_1 \dots b_m}{z_1^m}$. Hence $C_A = 1$. Furthermore, $\mathbf{b}$ must be a vee so we have $b_1 > b_2 > \cdots > b_m = 0$. □

Proof of Theorem 3A. Formulae (4.6) and (4.7) are equivalent by (3.6) and (4.5). To prove they give the Schubert polynomials, we take them for the moment as the definition of $\mathfrak{C}_w$ and show that $\mathfrak{C}_w$ satisfies the recurrence

$$\partial_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{w\sigma_i} & l(w\sigma_i) < l(w) \\ 0 & l(w\sigma_i) > l(w) \end{cases} \tag{4.36}$$

for all $i \ge 0$. For $i > 0$ we already have the recurrence by Lemma 4.3. Clearly, the constant term of $\mathfrak{C}_w$ is 0 if $w \neq 1$ and $\mathfrak{C}_1 = 1$.

It remains to prove (4.36) for $i = 0$. By Lemma 4.8,

$$\partial_0 \mathfrak{C}_w = \frac{1}{z_1} \sum_{\substack{\mathbf{abc} \in R(w) \\ b_k = 0}}^{\sim} 2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \begin{bmatrix} b_1 b_2 \dots b_{k-1} \\ z_1^{k-1} \end{bmatrix} \binom{b_k}{z_1} \begin{bmatrix} \mathbf{c} \\ z_2^{\gamma_2} z_3^{\gamma_3} \dots \end{bmatrix}. \tag{4.37}$$

The admissibility of the monomial $z_2^{\gamma_2} z_3^{\gamma_3} \cdots$ implies each letter $c_i > 1$, hence $\sigma_0 \sigma_{c_1} \cdots \sigma_{c_m} = \sigma_{c_1} \cdots \sigma_{c_m} \sigma_0$. Hence, (4.37) is equal to

$$\frac{1}{z_1} \sum_{\mathbf{abc}0 \in R(w)}^{\sim} 2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \begin{bmatrix} b_1 b_2 \dots b_{k-1} \\ z_1^{k-1} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ z_2^{\gamma_2} z_3^{\gamma_3} \dots \end{bmatrix} \binom{0}{z_1}. \tag{4.38}$$

If $l(w\sigma_0) > l(w)$ the summation is empty, while if $l(w\sigma_0) < l(w)$ it becomes

$$\sum_{\mathbf{abc} \in R(w\sigma_0)}^{\sim} 2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \begin{bmatrix} b_1 b_2 \dots b_{k-1} \\ z_1^{k-1} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ z_2^{\gamma_2} z_3^{\gamma_3} \dots \end{bmatrix}, \tag{4.39}$$

which is $\mathfrak{C}_{w\sigma_0}$. □

The Schubert polynomials of type B are defined by $\mathfrak{B}_w = 2^{-s(w)} \mathfrak{C}_w$. Since every reduced word for $w \in B_\infty$ contains $s(w)$ 0's, it is easy to see that the polynomials $\mathfrak{B}_w$ satisfy (4.36) with ∂_0^B in place of ∂_0.

We turn now to type D Schubert polynomials. Reduced words for elements of D_∞ use the alphabet $\{\hat{1}, 1, 2, \dots\}$. Our notation $\left[{\mathbf{c} \atop z^\gamma} \right]$ does not allow $c_i = \hat{1}$. Let us introduce a second notation $\left[{\mathbf{c} \atop z^\gamma} \right]^\wedge$ which allows $c_i = \hat{1}$ but not $c_i = 1$, and requires $z^\gamma \in \mathcal{A}_z(\hat{\mathbf{c}})$ where $\hat{\mathbf{c}}$ is the

word $\mathbf{c}$ with 1's and $\hat{1}$'s interchanged. Note that $\begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix}^\wedge$ is a reduced word admissible monomial pair if and only if $\begin{bmatrix}\hat{\mathbf{c}}\\ z^\gamma\end{bmatrix}$ is.

Lemma 4.9. *Let $\tilde{\mathfrak{D}}_w$ denote the polynomial defined by* (4.8) *and* (4.9). *For all $w \in D_\infty$,*

$$\sigma_0\tilde{\mathfrak{D}}_w = \sum_{\mathbf{abc}\in R(w)}{}^{\sim} 2^{i(\alpha)-o(\mathbf{a})}\binom{\mathbf{a}}{x^\alpha}\begin{bmatrix}b_1\dots b_k\\ z_1^k\end{bmatrix}^\wedge\begin{bmatrix}\mathbf{c}\\ z_2^{\gamma_2}z_3^{\gamma_3}\dots\end{bmatrix}. \tag{4.40}$$

Proof. We have

$$\sigma_0\tilde{\mathfrak{D}}_w = \sum_{\substack{uv=w\\ l(u)+l(v)=l(w)\\ v\in S_\infty}}\ \sum_{\mathbf{c}\in R(v)}{}^{\sim}\sigma_0\Big(E_u(X)\begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix}\Big) \tag{4.41}$$

$$= \sum_{\substack{uv=w\\ l(u)+l(v)=l(w)\\ v\in S_\infty}}\ \sum_{\mathbf{c}\in R(v)}{}^{\sim}(-1)^{\gamma_1}E_u(z_1+X)\begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix} \tag{4.42}$$

by Lemma 4.4. Expanding $E_u(z_1+X)$ in monomials by Proposition 3.10 we get

(4.43)

$$\begin{aligned}\sigma_0\tilde{\mathfrak{D}}_w &= \sum_{\mathbf{abc}\in R(w)}{}^{\sim}(-1)^{\gamma_1}2^{i(\alpha)-o(\mathbf{a})+\chi_k-o(\mathbf{b})}\binom{\mathbf{a}}{x^\alpha}\binom{\mathbf{b}}{z_1^k}\begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix}\\ &= \sum_{\mathbf{abc}'\in R(w)}{}^{\sim}(-1)^{\gamma_1}2^{i(\alpha)-o(\mathbf{a})+\chi_k-o(b_1\dots b_k)}\binom{\mathbf{a}}{x^\alpha}\binom{b_1\dots b_k}{z_1^k}\begin{bmatrix}b_{k+1}\dots b_m\\ z_1^{\gamma_1}\end{bmatrix}\begin{bmatrix}\mathbf{c}'\\ z_2^{\gamma_2}z_3^{\gamma_3}\dots\end{bmatrix},\end{aligned}$$

where $m=\gamma_1+k$ and $\chi_k=1$ if $k>0$ and 0 otherwise.

We need to determine the coefficient C_A of the general term of (4.43)

$$A = 2^{i(\alpha)-o(\mathbf{a})}\binom{\mathbf{a}}{x^\alpha}\left\langle\begin{matrix}b_1\dots b_m\\ z_1^m\end{matrix}\right\rangle\begin{bmatrix}c\\ z_2^{\gamma_2}z_3^{\gamma_3}\dots\end{bmatrix}. \tag{4.44}$$

There is a contribution of $2^{\chi_k-o(b_1\dots b_k)}(-1)^{m-k}$ to C_A for each k such that

$$\left\langle\begin{matrix}b_1\dots b_m\\ z_1^m\end{matrix}\right\rangle = \binom{b_1\dots b_k}{z_1^k}\begin{bmatrix}b_{k+1}\dots b_m\\ z_1^{k+1}\end{bmatrix}, \tag{4.45}$$

i.e., such that $b_1\dots b_k$ is a vee, $b_{k+1}>\dots>b_m$, and $b_m\neq\hat{1}$ unless $k=m$.

Case 1: $b_m\neq 1$ or $\hat{1}$

First, assume there exists an index i such that $b_i<b_{i+1}$ and choose i to be as large as possible. Then there are two possibilities for k in (4.45), namely $k=i$ and $k=i+1$. Therefore, $C_A = 2^{1-o(b_1\dots b_i)}\Big[(-1)^{m-i}+(-1)^{m-i-1}\Big]=0$.

Otherwise, $b_1 > b_2 > \cdots > b_m > 1$. For each $0 \le k \le m$, there is a contribution to C_A; $k = 0$ contributes $(-1)^m$ and each $0 < k \le m$ contributes $2(-1)^{m-k}$. Thus, $C_A = (-1)^m + \sum_{k=1}^{m} 2(-1)^{m-k} = 1$.

Case 2: $b_{m-1}b_m = \hat{1}1$ *or* $1\hat{1}$

These terms come in pairs since σ_1 and $\sigma_{\hat{1}}$ commute. For $b_{m-1}b_m = \hat{1}1$, there are two possibilities, $k = m-1$ and $k = m$, giving $C_A = -\frac{1}{2}$. For $b_{m-1}b_m = 1\hat{1}$ we must have $k = m$, giving $C_A = \frac{1}{2}$. Both terms have the same underlying monomial so their net contribution to $\sigma_0\tilde{\mathfrak{D}}_w$ is zero.

Case 3: $b_m = 1$ *and* $b_{m-1} \neq \hat{1}$

If $b_1 \dots b_{m-1}$ has an ascent, say $b_i < b_{i+1}$, then $C_A = 0$ as in Case 1. Otherwise, if $b_1 > b_2 > \cdots > b_{m-1} > 1$ then for $k = 0$, $k = m$, and $0 < k < m$ there are contributions of $(-1)^m, 1$ and $2(-1)^{m-k}$ respectively. Hence, $C_A = (-1)^m + 1 + \sum_{k=1}^{m-1} 2(-1)^{m-k} = 0$.

Case 4: $b_m = \hat{1}$ *and* $b_{m-1} \neq 1$

For this case, we must have $k = m$. We must also have $b_1 > b_2 > \cdots > b_{m-1}$ since $\mathbf{b} = b_1 \dots b_m$ must be a vee and $b_m = \hat{1}$ is its least element. Therefore $o(\mathbf{b}) = 1$ and $C_A = 1$.

Summarizing, there is a coefficient $C_A = 1$ for each A with $b_1 > \cdots > b_m$ and $b_m \neq 1$, and there is a net contribution of zero from all other terms. The terms with $C_A = 1$ are precisely those of the form

$$A = 2^{i(\alpha)-o(\mathbf{a})} \begin{pmatrix} \mathbf{a} \\ x^\alpha \end{pmatrix} \begin{bmatrix} b_1 \dots b_m \\ z_1^m \end{bmatrix}^{\wedge} \begin{bmatrix} c \\ z_2^{\gamma_2} z_3^{\gamma_3} \dots \end{bmatrix}, \tag{4.46}$$

proving (4.40). □

Corollary 4.10. *For all* $w \in D_\infty$,

$$\sigma_0\tilde{\mathfrak{D}}_w = \tilde{\mathfrak{D}}_{\widehat{w}}, \tag{4.47}$$

where $\widehat{w}$ *is the image of* w *under the involution of* D_∞ *given by interchanging* σ_1 *and* $\sigma_{\hat{1}}$.

Proof. By Lemma 4.9,

$$\sigma_0\tilde{\mathfrak{D}}_w = \sum_{\mathbf{abc}\in R(w)}^{\sim} 2^{i(\alpha)-o(\mathbf{a})} \begin{pmatrix} \mathbf{a} \\ x^\alpha \end{pmatrix} \begin{bmatrix} \mathbf{b} \\ z_1^k \end{bmatrix}^{\wedge} \begin{bmatrix} \mathbf{c} \\ z_2^{\gamma_2} z_{\gamma_3}^3 \dots \end{bmatrix} \tag{4.48}$$

$$= \sum_{\widehat{\mathbf{a}}\widehat{\mathbf{b}}\widehat{\mathbf{c}}\in R(\widehat{w})}^{\sim} 2^{i(\alpha)-o(\widehat{\mathbf{a}})} \begin{pmatrix} \widehat{\mathbf{a}} \\ x^\alpha \end{pmatrix} \begin{bmatrix} \widehat{\mathbf{b}} \\ z_1^k \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{c}} \\ z_2^{\gamma_2} z_{\gamma_3}^3 \dots \end{bmatrix} \tag{4.49}$$

$$= \tilde{\mathfrak{D}}_{\widehat{w}}. \quad \square \tag{4.50}$$

Proof of Theorem 4A. Formulas (4.8) and (4.9) are equivalent by Proposition 3.10 and Proposition 4.1. To prove they give Schubert polynomials of type D, we take them for

the moment as the definition of $\mathfrak{D}_w$ and show that $\mathfrak{D}_w$ then satisfies the recurrence

$$\partial_i \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{w\sigma_i} & l(w\sigma_i) < l(w) \\ 0 & l(w\sigma_i) > l(w) \end{cases} \tag{4.51}$$

for all $i \in \{\hat{1}, 1, 2, \dots\}$. For $i \neq \hat{1}$ we already have the recurrence by Lemma 4.3. The constant term of $\mathfrak{D}_w$ is 0 if $w \neq 1$ and $\mathfrak{D}_1 = 1$.

It remains to prove (4.51) for $i = \hat{1}$. We shall take advantage of the symmetry between the generators σ_1 and $\sigma_{\hat{1}}$. A simple computation shows

$$\sigma_0 \partial_1 \sigma_0 f = \partial_{\hat{1}} f = \frac{f - \sigma_{\hat{1}} f}{-z_1 - z_2}. \tag{4.52}$$

Therefore, by repeated use of Corollary 4.10 and Lemma 4.3,

$$\begin{aligned} \partial_{\hat{1}} \mathfrak{D}_w &= \sigma_0 \partial_1 \sigma_0 \mathfrak{D}_w \\ &= \sigma_0 \partial_1 \mathfrak{D}_{\widehat{w}} \\ &= \sigma_0 \begin{cases} \mathfrak{D}_{\widehat{w}\sigma_1} & l(\widehat{w}\sigma_1) < l(\widehat{w}) \\ 0 & l(\widehat{w}\sigma_1) > l(\widehat{w}) \end{cases} \\ &= \begin{cases} \mathfrak{D}_{w\sigma_{\widehat{1}}} & l(w\sigma_{\hat{1}}) < l(w) \\ 0 & l(w\sigma_{\hat{1}}) > l(w). \end{cases} \quad \square \end{aligned} \tag{4.53}$$

Theorem 3B. *Given a partition μ with distinct parts, let $w = \overline{\mu_1}\ \overline{\mu_2} \dots \overline{\mu_l} 12 \dots$. Then we have*

$$\mathfrak{C}_w = Q_\mu(X), \quad \mathfrak{B}_w = P_\mu(X). \tag{4.54}$$

Proof. Given $w = \overline{\mu_1}\ \overline{\mu_2} \dots \overline{\mu_l} 12 \dots$, the only element $v \in S_\infty$ such that $uv = w$ and $l(u) + l(v) = l(w)$ is $v = 1$. Therefore $\mathfrak{C}_w = F_w$, and by Proposition 3.14 $F_w = Q_\mu$. By definition $\mathfrak{B}_w = 2^{-s(w)} \mathfrak{C}_w$ where $s(w)$ is the number of signs changed by w. Hence, $\mathfrak{B}_w = 2^{-l} Q_\mu = P_\mu$. $\square$

Theorem 4B. *Given a partition μ with distinct parts, let $\nu_i = 1 + \mu_i$, taking $\mu_l = 0$ if necessary to make the number of parts even. Then for $w = \overline{\nu_1}\ \overline{\nu_2} \dots \overline{\nu_l} 12 \dots$, we have*

$$\mathfrak{D}_w = P_\mu(X). \tag{4.55}$$

Proof. As in the previous theorem, $\mathfrak{D}_w = E_w$, and $E_w = P_\mu$ by Proposition 3.13. $\square$

Next we show that the polynomials $\mathfrak{B}_w$, $\mathfrak{C}_w$ and $\mathfrak{D}_w$ are integral bases of the rings in which they lie. We do this by identifying their leading terms with respect to an appropriate ordering.

Definition. Given two shifted shapes λ and μ and two compositions $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ with $m = \sum \alpha_i$ and $n = \sum \beta_j$, we say $z^\alpha Q_\lambda < z^\beta Q_\mu$ if any of the following hold:

(1) $m < n$.
(2) $m = n$ and $\alpha <_r \beta$ in reverse lexicographic order.
(3) $\alpha = \beta$ and $\lambda < \mu$ in an arbitrarily chosen total ordering.

Reverse lexicographic order means $\alpha <_r \beta$ if $\alpha_i < \beta_i$ for some i and $\alpha_l = \beta_l$ for all $l > i$.

Definition. [**13**] Given $w \in S_n$, for each $i \geq 1$ let $c_i(w) = |\{j \mid j > i \text{ and } w(j) < w(i)\}|$. The composition

$$c(w) = \Big(c_1(w), c_2(w), \dots, c_n(w)\Big) \tag{4.56}$$

is the *code* of w.

Lemma 4.11. *Under the ordering $<$, the leading term of $\mathfrak{S}_w$ is distinct for each $w \in S_\infty$ and is given by $z^{c(w)}$.*

Proof. The lemma follows by induction from the transition equation for Schubert polynomials of type A, formula (4.16) of [**13**]. □

Lemma 4.12. *For every monomial $z^\alpha Q_\mu$ there is a unique $w \in B_\infty$ such that $z^\alpha Q_\mu$ is the leading term of $\mathfrak{C}_w$ under the ordering $<$ defined above. For this same w, $z^\alpha P_\mu$ is the leading term of $\mathfrak{B}_w$.*

Proof. Let $w = w(1)w(2)\dots w(n)$ in one line notation. Let u_w be the increasing arrangement of the numerals $w(1), w(2), \dots w(n)$, and let $v_w = u_w^{-1} w$. Then $l(u_w) + l(v_w) = l(w)$ and $l(v_w) > l(v)$ for any other $v \in S_\infty$ such that $uv = w$ and $l(u) + l(v) = l(w)$. Therefore, the leading term of $\mathfrak{C}_w$ comes from the expansion of $F_{u_w} \mathfrak{S}_{v_w}$, by Theorem 3A.

Let μ_w be the shape such that $\mathfrak{C}_{u_w} = F_{u_w} = Q_{\mu_w}$. This shape μ_w exists by Theorem 3B. By Lemma 4.11, $z^{c(v_w)}$ is the leading term of $\mathfrak{S}_{v_w}$. Therefore, $z^{c(v_w)} Q_{\mu_w}$ is the leading term of $\mathfrak{C}_w$.

Given any $z^\alpha Q_\mu$, let $v \in S_\infty$ be the unique permutation such that $c(v) = \alpha$. Define $u \in B_\infty$ by $u = \overline{\mu_1}\ \overline{\mu_2} \dots \overline{\mu_l} 12 \dots$. Then for $w = uv \in B_n$, $u_w = u$ and $v_w = v$, so $\mathfrak{C}_w$ has $z^\alpha Q_\mu$ as its leading term. This w is unique since μ determines u_w and α determines v_w.

From the description of u we see that $s(w) = l(\mu)$, so the leading term of $\mathfrak{B}_w = 2^{-s(w)} \mathfrak{C}_{\mathfrak{w}}$ is $z^\alpha P_\mu$. □

Lemma 4.13. *For every monomial $z^\alpha Q_\mu$ there is a unique $w \in D_\infty$ such that $z^\alpha Q_\mu$ is the leading term of $\mathfrak{D}_w$ under the ordering $>$ defined above.*

Proof. The only difference between this proof and the previous one is the computation of the leading term. Given $w = w(1) \dots w(n) \in D_n$, again let u_w be the increasing rearrangement of the $w(i)$, so $u_w = \overline{\nu_1}\ \overline{\nu_2} \dots \overline{\nu_l} 12 \dots$ for some partition ν. Let $v_w = u_w^{-1} w$. The leading term of $\mathfrak{D}_w$ is $z^{c(v_w)} P_\mu$ where $\mu = (\nu_1 - 1, \nu_2 - 1, \dots, \nu_l - 1)$. □

Lemma 4.14. *The Schubert polynomials $\mathfrak{B}_w$ lie in the ring $\mathbb{Z}[z_1, z_2, \ldots ; P_\mu]$.*

Proof. Consider a general term $\binom{\mathbf{a}}{x^\alpha}\binom{\mathbf{b}}{z^\gamma}$ occurring in $\mathfrak{C}_w$, where $\mathbf{ab} \in R(w)$. There are $s(w)$ 0's in $\mathbf{a}$, with at least one peak between each consecutive pair of them. This forces x^α to contain at least $s(w)$ distinct variables with non-zero exponent.

Every term $z^\alpha Q_\mu$ occuring in $\mathfrak{C}_w$ has positive coefficient, so no monomials cancel among terms. In particular, $Q_\mu(X)$ cannot contain any monomial involving fewer than $s(w)$ distinct variables. This forces $l(\mu) \geq s(w)$, and hence the corresponding term in $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$ is an integral multiple of $z^\alpha P_\mu$. □

Theorem 3C. *The Schubert polynomials $\mathfrak{C}_w$ of type C are a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[z_1, z_2, \ldots ; Q_\mu]$. The polynomials $\mathfrak{B}_w$ are a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[z_1, z_2, \ldots ; P_\mu]$.*

Proof. By Proposition 3.1, the sets $\{z^\alpha Q_\mu\}$ and $\{z^\alpha P_\mu\}$ are $\mathbb{Z}$-bases for the rings $\mathbb{Z}[z_1, z_2, \ldots ; Q_\mu]$ and $\mathbb{Z}[z_1, z_2, \ldots ; P_\mu]$, respectively.

Since the $\mathfrak{C}_w$ have distinct leading terms, they are linearly independent. They span the ring $\mathbb{Z}[z_1, z_2, \ldots ; Q_\mu]$ since every monomial $z^\alpha Q_\mu$ occurs as the leading term of some $\mathfrak{C}_w$. Analogous remarks apply to the $\mathfrak{B}_w$. □

Theorem 4C. *The Schubert polynomials of type D are a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[z_1, z_2, \ldots ; P_\mu]$.*

Proof. Same as the preceding proof. □

The formulas we have given for Schubert polynomials of types B, C, and D, though fully explicit, are ill-suited to practical computation because of the difficulty of using the Edelman–Greene correspondences to evaluate $F_u(X)$ and $E_u(X)$. An alternative method is to compute Schubert polynomials by applying iterated divided difference operators to the 'top' polynomials $\mathfrak{C}_{w_0^B}$ and $\mathfrak{D}_{w_0^D}$. This method is facilitated by the use of Corollary 4.5, together with convenient expressions for $\mathfrak{C}_{w_0^B}$ and $\mathfrak{D}_{w_0^D}$ which we now derive.

Definition. Let $\lambda \subseteq \mu$ be partitions of length at most k. The corresponding *skew multi-Schur function* is defined by

(4.57)
$$S_{\mu/\lambda}(z_1, z_1 + z_2, \ldots , z_1 + z_2 + \cdots + z_k) = \det \left[h_{\mu_i - \lambda_j + j - i}(z_1, z_2 \ldots , z_i) \right]_{i,j=1}^{k}$$

(4.58)
$$= \sum_T z^T,$$

where h_m denotes the complete homogeneous symmetric function of degree m, and T ranges over column-strict tableaux of shape μ/λ in which entries in row i do not exceed i.

The equivalence of formulas (4.57) and (4.58) is due to Gessel [**8**]—see also [**19**], since [**8**] is unpublished.

Proposition 4.15. *Let w_0^B, w_0^D denote the longest elements in B_n and D_n respectively. Let $\delta_k = (k, k-1, \ldots, 1)$. Then*

$$\mathfrak{C}_{w_0^B} = \sum_\lambda Q_{\delta_n+\lambda}(X) S_{\delta_{n-1}/\lambda'}(z_1, z_1+z_2, \ldots, z_1+z_2+\cdots+z_{n-1}), \tag{4.59}$$

$$\mathfrak{D}_{w_0^D} = \sum_\lambda P_{\delta_{n-1}+\lambda}(X) S_{\delta_{n-1}/\lambda'}(z_1, z_1+z_2, \ldots, z_1+z_2+\cdots+z_{n-1}). \tag{4.60}$$

Here λ' denotes the partition conjugate to λ.

Proof. For every $v \in S_n$, we have $l(w_0^B v^{-1}) + l(v) = l(w_0^B)$ and $l(w_0^D v^{-1}) + l(v) = l(w_0^D)$. Hence by Proposition 3.16,

$$\mathfrak{C}_{w_0^B} = \sum_{v\in S_n} F_{w_0^B v^{-1}}(X)\mathfrak{S}_v(Z) = \sum_{v\in S_n}\sum_\lambda g^\lambda_{v_0 v^{-1}} Q_{\delta_n+\lambda}(X)\mathfrak{S}_v(Z), \tag{4.61}$$

$$\mathfrak{D}_{w_0^D} = \sum_{v\in S_n} E_{w_0^D v^{-1}}(X)\mathfrak{S}_v(Z) = \sum_{v\in S_n}\sum_\lambda g^\lambda_{v_0 v^{-1}} P_{\delta_{n-1}+\lambda}(X)\mathfrak{S}_v(Z). \tag{4.62}$$

It remains to prove, for each λ,

$$\sum_{v\in S_n} g^\lambda_{v_0 v^{-1}}\mathfrak{S}_v(Z) = S_{\delta_{n-1}/\lambda'}(z_1, z_1+z_2, \ldots, z_1+z_2+\cdots+z_{n-1}). \tag{4.63}$$

Equations (4.9) and (7.14) of [**13**] show that

$$S_{\delta_{n-1}}(Y+z_1, Y+z_1+z_2, \ldots, Y+z_1+z_2+\cdots+z_{n-1}) \tag{4.64}$$

$$= \mathfrak{S}_{1^m\times v_0}(y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_{n-1}) \tag{4.65}$$

$$= \sum_{v\in S_n}\sum_\lambda g^\lambda_{v v_0} s_\lambda(Y)\mathfrak{S}_v(Z), \tag{4.66}$$

where $Y = y_1 + y_2 + \cdots + y_m$. Using the identity $g^\lambda_w = g^{\lambda'}_{w^{-1}}$ of [**5**] and replacing λ by its conjugate in the summation, the last expression becomes

$$\sum_{v\in S_n}\sum_\lambda g^\lambda_{v_0 v^{-1}} s_{\lambda'}(Y)\mathfrak{S}_v(Z). \tag{4.67}$$

We also have by a general identity for skew multi-Schur functions

$$\begin{aligned} &S_{\delta_{n-1}}(Y+z_1, Y+z_1+z_2, \ldots, Y+z_1+z_2+\cdots+z_{n-1}) \\ &\qquad = \sum_\lambda s_\lambda(Y) S_{\delta_{n-1}/\lambda}(z_1, z_1+z_2, \ldots, z_1+z_2+\cdots z_{n-1}). \end{aligned} \tag{4.68}$$

Equating coefficients of $s_{\lambda'}(Y)$ in (4.67) and (4.68) gives (4.63). □

w	$\mathfrak{B}_w$
$1\,2\,3{=}1$	1
$\overline{1}\,2\,3{=}\sigma_0$	P_1
$2\,1\,3{=}\sigma_1$	$2\,P_1 + z_1$
$\overline{2}\,1\,3{=}\sigma_1\sigma_0$	P_2
$2\,\overline{1}\,3{=}\sigma_0\sigma_1$	$P_2 + P_1\,z_1$
$\overline{2}\,\overline{1}\,3{=}\sigma_0\sigma_1\sigma_0$	P_{21}
$1\,\overline{2}\,3{=}\sigma_1\sigma_0\sigma_1$	$P_3 + P_2\,z_1$
$\overline{1}\,\overline{2}\,3{=}\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{31} + P_{21}\,z_1$
$1\,3\,2{=}\sigma_2$	$2\,P_1 + z_1 + z_2$
$\overline{1}\,3\,2{=}\sigma_2\sigma_0$	$2\,P_2 + P_1\,z_1 + P_1\,z_2$
$3\,1\,2{=}\sigma_2\sigma_1$	$2\,P_2 + 2\,P_1\,z_1 + {z_1}^2$
$\overline{3}\,1\,2{=}\sigma_2\sigma_1\sigma_0$	P_3
$3\,\overline{1}\,2{=}\sigma_2\sigma_0\sigma_1$	$P_3 + 2\,P_{21} + 2\,P_2\,z_1 + P_1\,{z_1}^2$
$\overline{3}\,\overline{1}\,2{=}\sigma_2\sigma_0\sigma_1\sigma_0$	P_{31}
$1\,\overline{3}\,2{=}\sigma_2\sigma_1\sigma_0\sigma_1$	$P_4 + P_3\,z_1$
$\overline{1}\,\overline{3}\,2{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{41} + P_{31}\,z_1$
$2\,3\,1{=}\sigma_1\sigma_2$	$2\,P_2 + 2\,P_1\,z_1 + 2\,P_1\,z_2 + z_1\,z_2$
$\overline{2}\,3\,1{=}\sigma_1\sigma_2\sigma_0$	$P_3 + 2\,P_{21} + P_2\,z_1 + P_2\,z_2$
$3\,2\,1{=}\sigma_1\sigma_2\sigma_1$	$2\,P_3 + 4\,P_{21} + 4\,P_2\,z_1 + 2\,P_1\,{z_1}^2 + 2\,P_2\,z_2 + 2\,P_1\,z_1\,z_2 + {z_1}^2\,z_2$
$\overline{3}\,2\,1{=}\sigma_1\sigma_2\sigma_1\sigma_0$	$P_4 + 2\,P_{31} + P_3\,z_1 + P_3\,z_2$
$3\,\overline{2}\,1{=}\sigma_1\sigma_2\sigma_0\sigma_1$	$2\,P_{31} + P_3\,z_1 + 2\,P_{21}\,z_1 + P_2\,{z_1}^2$
$\overline{3}\,\overline{2}\,1{=}\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	P_{32}
$2\,\overline{3}\,1{=}\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$2\,P_{41} + P_4\,z_1 + 2\,P_{31}\,z_1 + P_3\,{z_1}^2$
$\overline{2}\,\overline{3}\,1{=}\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{42} + P_{32}\,z_1$
$2\,3\,\overline{1}{=}\sigma_0\sigma_1\sigma_2$	$P_3 + P_2\,z_1 + P_2\,z_2 + P_1\,z_1\,z_2$
$\overline{2}\,3\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_0$	$P_{31} + P_{21}\,z_1 + P_{21}\,z_2$
$3\,2\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_1$	$P_4 + 2\,P_{31} + 2\,P_3\,z_1 + 2\,P_{21}\,z_1 + P_2\,{z_1}^2 + P_3\,z_2 + 2\,P_{21}\,z_2 + 2\,P_2\,z_1\,z_2 + P_1\,{z_1}^2\,z_2$
$\overline{3}\,2\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$P_{32} + P_{41} + P_{31}\,z_1 + P_{31}\,z_2$
$3\,\overline{2}\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$P_{32} + P_{31}\,z_1 + P_{21}\,{z_1}^2$
$\overline{3}\,\overline{2}\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	P_{321}
$2\,\overline{3}\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$P_{42} + P_{32}\,z_1 + P_{41}\,z_1 + P_{31}\,{z_1}^2$
$\overline{2}\,\overline{3}\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{421} + P_{321}\,z_1$
$1\,3\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2$	$P_4 + P_3\,z_1 + P_3\,z_2 + P_2\,z_1\,z_2$
$\overline{1}\,3\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0$	$P_{41} + P_{31}\,z_1 + P_{31}\,z_2 + P_{21}\,z_1\,z_2$
$3\,1\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1$	$2\,P_{41} + P_4\,z_1 + 2\,P_{31}\,z_1 + P_3\,{z_1}^2 + 2\,P_{31}\,z_2 + P_3\,z_1\,z_2 + 2\,P_{21}\,z_1\,z_2 + P_2\,{z_1}^2\,z_2$
$\overline{3}\,1\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$P_{42} + P_{32}\,z_1 + P_{32}\,z_2$
$3\,\overline{1}\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$P_{42} + P_{32}\,z_1 + P_{41}\,z_1 + P_{31}\,{z_1}^2 + P_{32}\,z_2 + P_{31}\,z_1\,z_2 + P_{21}\,{z_1}^2\,z_2$
$\overline{3}\,\overline{1}\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	$P_{421} + P_{321}\,z_1 + P_{321}\,z_2$
$1\,\overline{3}\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$P_{43} + P_{42}\,z_1 + P_{32}\,{z_1}^2$
$\overline{1}\,\overline{3}\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{431} + P_{421}\,z_1 + P_{321}\,{z_1}^2$
$1\,2\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2$	$P_5 + P_4\,z_1 + P_4\,z_2 + P_3\,z_1\,z_2$
$\overline{1}\,2\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0$	$P_{51} + P_{41}\,z_1 + P_{41}\,z_2 + P_{31}\,z_1\,z_2$
$2\,1\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1$	$2\,P_{51} + P_5\,z_1 + 2\,P_{41}\,z_1 + P_4\,{z_1}^2 + 2\,P_{41}\,z_2 + P_4\,z_1\,z_2 + 2\,P_{31}\,z_1\,z_2 + P_3\,{z_1}^2\,z_2$
$\overline{2}\,1\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$P_{52} + P_{42}\,z_1 + P_{42}\,z_2 + P_{32}\,z_1\,z_2$
$2\,\overline{1}\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$P_{52} + P_{42}\,z_1 + P_{51}\,z_1 + P_{41}\,{z_1}^2 + P_{42}\,z_2 + P_{32}\,z_1\,z_2 + P_{41}\,z_1\,z_2 + P_{31}\,{z_1}^2\,z_2$
$\overline{2}\,\overline{1}\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	$P_{521} + P_{421}\,z_1 + P_{421}\,z_2 + P_{321}\,z_1\,z_2$
$1\,\overline{2}\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$P_{53} + P_{43}\,z_1 + P_{52}\,z_1 + P_{42}\,{z_1}^2 + P_{43}\,z_2 + P_{42}\,z_1\,z_2 + P_{32}\,{z_1}^2\,z_2$
$\overline{1}\,\overline{2}\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{531} + P_{431}\,z_1 + P_{521}\,z_1 + P_{421}\,{z_1}^2 + P_{431}\,z_2 + P_{421}\,z_1\,z_2 + P_{321}\,{z_1}^2\,z_2$

TABLE 1. Type B Schubert polynomials for $w \in B_3$

w	$\mathfrak{C}_w$
$1\,2\,3{=}1$	1
$\overline{1}\,2\,3{=}\sigma_0$	Q_1
$2\,1\,3{=}\sigma_1$	$Q_1 + z_1$
$\overline{2}\,1\,3{=}\sigma_1\sigma_0$	Q_2
$2\,\overline{1}\,3{=}\sigma_0\sigma_1$	$Q_2 + Q_1\,z_1$
$\overline{2}\,\overline{1}\,3{=}\sigma_0\sigma_1\sigma_0$	Q_{21}
$1\,\overline{2}\,3{=}\sigma_1\sigma_0\sigma_1$	$Q_3 + Q_2\,z_1$
$\overline{1}\,\overline{2}\,3{=}\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{31} + Q_{21}\,z_1$
$1\,3\,2{=}\sigma_2$	$Q_1 + z_1 + z_2$
$\overline{1}\,3\,2{=}\sigma_2\sigma_0$	$2\,Q_2 + Q_1\,z_1 + Q_1\,z_2$
$3\,1\,2{=}\sigma_2\sigma_1$	$Q_2 + Q_1\,z_1 + {z_1}^2$
$\overline{3}\,1\,2{=}\sigma_2\sigma_1\sigma_0$	Q_3
$3\,\overline{1}\,2{=}\sigma_2\sigma_0\sigma_1$	$Q_3 + Q_{21} + 2\,Q_2\,z_1 + Q_1\,{z_1}^2$
$\overline{3}\,\overline{1}\,2{=}\sigma_2\sigma_0\sigma_1\sigma_0$	Q_{31}
$1\,\overline{3}\,2{=}\sigma_2\sigma_1\sigma_0\sigma_1$	$Q_4 + Q_3\,z_1$
$\overline{1}\,\overline{3}\,2{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{41} + Q_{31}\,z_1$
$2\,3\,1{=}\sigma_1\sigma_2$	$Q_2 + Q_1\,z_1 + Q_1\,z_2 + z_1\,z_2$
$\overline{2}\,3\,1{=}\sigma_1\sigma_2\sigma_0$	$Q_3 + Q_{21} + Q_2\,z_1 + Q_2\,z_2$
$3\,2\,1{=}\sigma_1\sigma_2\sigma_1$	$Q_3 + Q_{21} + 2\,Q_2\,z_1 + Q_1\,{z_1}^2 + Q_2\,z_2 + Q_1\,z_1\,z_2 + {z_1}^2\,z_2$
$\overline{3}\,2\,1{=}\sigma_1\sigma_2\sigma_1\sigma_0$	$Q_4 + Q_{31} + Q_3\,z_1 + Q_3\,z_2$
$3\,\overline{2}\,1{=}\sigma_1\sigma_2\sigma_0\sigma_1$	$Q_{31} + Q_3\,z_1 + Q_{21}\,z_1 + Q_2\,{z_1}^2$
$\overline{3}\,\overline{2}\,1{=}\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	Q_{32}
$2\,\overline{3}\,1{=}\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$Q_{41} + Q_4\,z_1 + Q_{31}\,z_1 + Q_3\,{z_1}^2$
$\overline{2}\,\overline{3}\,1{=}\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{42} + Q_{32}\,z_1$
$2\,3\,\overline{1}{=}\sigma_0\sigma_1\sigma_2$	$Q_3 + Q_2\,z_1 + Q_2\,z_2 + Q_1\,z_1\,z_2$
$\overline{2}\,3\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_0$	$Q_{31} + Q_{21}\,z_1 + Q_{21}\,z_2$
$3\,2\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_1$	$Q_4 + Q_{31} + 2\,Q_3\,z_1 + Q_{21}\,z_1 + Q_2\,{z_1}^2 + Q_3\,z_2 + Q_{21}\,z_2 + 2\,Q_2\,z_1\,z_2 + Q_1\,{z_1}^2\,z_2$
$\overline{3}\,2\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$Q_{32} + Q_{41} + Q_{31}\,z_1 + Q_{31}\,z_2$
$3\,\overline{2}\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$Q_{32} + Q_{31}\,z_1 + Q_{21}\,{z_1}^2$
$\overline{3}\,\overline{2}\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	Q_{321}
$2\,\overline{3}\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$Q_{42} + Q_{32}\,z_1 + Q_{41}\,z_1 + Q_{31}\,{z_1}^2$
$\overline{2}\,\overline{3}\,\overline{1}{=}\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{421} + Q_{321}\,z_1$
$1\,3\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2$	$Q_4 + Q_3\,z_1 + Q_3\,z_2 + Q_2\,z_1\,z_2$
$\overline{1}\,3\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0$	$Q_{41} + Q_{31}\,z_1 + Q_{31}\,z_2 + Q_{21}\,z_1\,z_2$
$3\,1\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1$	$Q_{41} + Q_4\,z_1 + Q_{31}\,z_1 + Q_3\,{z_1}^2 + Q_{31}\,z_2 + Q_3\,z_1\,z_2 + Q_{21}\,z_1\,z_2 + Q_2\,{z_1}^2\,z_2$
$\overline{3}\,1\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$Q_{42} + Q_{32}\,z_1 + Q_{32}\,z_2$
$3\,\overline{1}\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$Q_{42} + Q_{32}\,z_1 + Q_{41}\,z_1 + Q_{31}\,{z_1}^2 + Q_{32}\,z_2 + Q_{31}\,z_1\,z_2 + Q_{21}\,{z_1}^2\,z_2$
$\overline{3}\,\overline{1}\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	$Q_{421} + Q_{321}\,z_1 + Q_{321}\,z_2$
$1\,\overline{3}\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$Q_{43} + Q_{42}\,z_1 + Q_{32}\,{z_1}^2$
$\overline{1}\,\overline{3}\,\overline{2}{=}\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{431} + Q_{421}\,z_1 + Q_{321}\,{z_1}^2$
$1\,2\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2$	$Q_5 + Q_4\,z_1 + Q_4\,z_2 + Q_3\,z_1\,z_2$
$\overline{1}\,2\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0$	$Q_{51} + Q_{41}\,z_1 + Q_{41}\,z_2 + Q_{31}\,z_1\,z_2$
$2\,1\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1$	$Q_{51} + Q_5\,z_1 + Q_{41}\,z_1 + Q_4\,{z_1}^2 + Q_{41}\,z_2 + Q_4\,z_1\,z_2 + Q_{31}\,z_1\,z_2 + Q_3\,{z_1}^2\,z_2$
$\overline{2}\,1\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$Q_{52} + Q_{42}\,z_1 + Q_{42}\,z_2 + Q_{32}\,z_1\,z_2$
$2\,\overline{1}\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$Q_{52} + Q_{42}\,z_1 + Q_{51}\,z_1 + Q_{41}\,{z_1}^2 + Q_{42}\,z_2 + Q_{32}\,z_1\,z_2 + Q_{41}\,z_1\,z_2 + Q_{31}\,{z_1}^2\,z_2$
$\overline{2}\,\overline{1}\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	$Q_{521} + Q_{421}\,z_1 + Q_{421}\,z_2 + Q_{321}\,z_1\,z_2$
$1\,\overline{2}\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$Q_{53} + Q_{43}\,z_1 + Q_{52}\,z_1 + Q_{42}\,{z_1}^2 + Q_{43}\,z_2 + Q_{42}\,z_1\,z_2 + Q_{32}\,{z_1}^2\,z_2$
$\overline{1}\,\overline{2}\,\overline{3}{=}\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{531} + Q_{431}\,z_1 + Q_{521}\,z_1 + Q_{421}\,{z_1}^2 + Q_{431}\,z_2 + Q_{421}\,z_1\,z_2 + Q_{321}\,{z_1}^2\,z_2$

TABLE 2. Type C Schubert polynomials for $w \in B_3$

w	$\mathfrak{D}_w$
$1\,2\,3{=}1$	1
$2\,1\,3{=}\sigma_1$	$P_1 + z_1$
$\overline{2}\,\overline{1}\,3{=}\sigma_{\hat{1}}$	P_1
$\overline{1}\,\overline{2}\,3{=}\sigma_1\sigma_{\hat{1}}$	$P_2 + P_1\,z_1$
$1\,3\,2{=}\sigma_2$	$2\,P_1 + z_1 + z_2$
$3\,1\,2{=}\sigma_2\sigma_1$	$P_2 + 2\,P_1\,z_1 + {z_1}^2$
$\overline{3}\,\overline{1}\,2{=}\sigma_2\sigma_{\hat{1}}$	P_2
$\overline{1}\,\overline{3}\,2{=}\sigma_2\sigma_1\sigma_{\hat{1}}$	$P_3 + P_2\,z_1$
$2\,3\,1{=}\sigma_1\sigma_2$	$P_2 + P_1\,z_1 + P_1\,z_2 + z_1\,z_2$
$3\,2\,1{=}\sigma_1\sigma_2\sigma_1$	$P_3 + P_{21} + 2\,P_2\,z_1 + P_1\,{z_1}^2 + P_2\,z_2 + 2\,P_1\,z_1\,z_2 + {z_1}^2\,z_2$
$\overline{3}\,\overline{2}\,1{=}\sigma_1\sigma_2\sigma_{\hat{1}}$	P_{21}
$\overline{2}\,\overline{3}\,1{=}\sigma_1\sigma_2\sigma_1\sigma_{\hat{1}}$	$P_{31} + P_{21}\,z_1$
$\overline{2}\,3\,\overline{1}{=}\sigma_{\hat{1}}\sigma_2$	$P_2 + P_1\,z_1 + P_1\,z_2$
$3\,\overline{2}\,\overline{1}{=}\sigma_{\hat{1}}\sigma_2\sigma_1$	$P_{21} + P_2\,z_1 + P_1\,{z_1}^2$
$\overline{3}\,2\,\overline{1}{=}\sigma_{\hat{1}}\sigma_2\sigma_{\hat{1}}$	$P_3 + P_{21} + P_2\,z_1 + P_2\,z_2$
$2\,\overline{3}\,\overline{1}{=}\sigma_{\hat{1}}\sigma_2\sigma_1\sigma_{\hat{1}}$	$P_{31} + P_3\,z_1 + P_{21}\,z_1 + P_2\,{z_1}^2$
$\overline{1}\,3\,\overline{2}{=}\sigma_1\sigma_{\hat{1}}\sigma_2$	$P_3 + P_2\,z_1 + P_2\,z_2 + P_1\,z_1\,z_2$
$3\,\overline{1}\,\overline{2}{=}\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_1$	$P_{31} + P_3\,z_1 + P_{21}\,z_1 + P_2\,{z_1}^2 + P_{21}\,z_2 + P_2\,z_1\,z_2 + P_1\,{z_1}^2\,z_2$
$\overline{3}\,1\,\overline{2}{=}\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_{\hat{1}}$	$P_{31} + P_{21}\,z_1 + P_{21}\,z_2$
$1\,\overline{3}\,\overline{2}{=}\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_1\sigma_{\hat{1}}$	$P_{32} + P_{31}\,z_1 + P_{21}\,{z_1}^2$
$\overline{1}\,2\,\overline{3}{=}\sigma_2\sigma_1\sigma_{\hat{1}}\sigma_2$	$P_4 + P_3\,z_1 + P_3\,z_2 + P_2\,z_1\,z_2$
$2\,\overline{1}\,\overline{3}{=}\sigma_2\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_1$	$P_{41} + P_4\,z_1 + P_{31}\,z_1 + P_3\,{z_1}^2 + P_{31}\,z_2 + P_3\,z_1\,z_2 + P_{21}\,z_1\,z_2 + P_2\,{z_1}^2\,z_2$
$\overline{2}\,1\,\overline{3}{=}\sigma_2\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_{\hat{1}}$	$P_{41} + P_{31}\,z_1 + P_{31}\,z_2 + P_{21}\,z_1\,z_2$
$1\,\overline{2}\,\overline{3}{=}\sigma_2\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_1\sigma_{\hat{1}}$	$P_{42} + P_{32}\,z_1 + P_{41}\,z_1 + P_{31}\,{z_1}^2 + P_{32}\,z_2 + P_{31}\,z_1\,z_2 + P_{21}\,{z_1}^2\,z_2$

TABLE 3. Type D Schubert polynomials for $w \in D_3$

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