Basic Schubert Calculus

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ICERM: Introductory Workshop:
Combinatorial Algebraic Geometry
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Outline

Enumerative Algebraic Geometry and Hilbert's 15th Problem

Introduction to Flag Manifolds and Schubert Varieties

Schubert Problems in Intersection Theory

Ancient Questions Enumerative Algebraic Geometry

1. How many points are in the intersection of two lines in $\mathbb{R}^2$? Ans: 0 or 1 or ∞.
2. Given 2 circles in the plane, how many common tangents do they have? Ans: 0,1,2,3,4,∞. Draw pictures!
3. Given 3 circles in the plane, how many circles are tangent to all 3?

Ancient Questions Enumerative Algebraic Geometry

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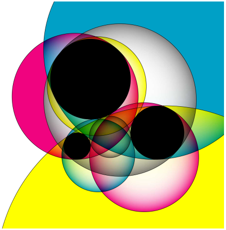
2. Given 2 circles in the plane, how many common tangents do they have? Ans: 0,1,2,3,4,∞. Draw pictures!
3. Given 3 circles in the plane, how many circles are tangent to all 3? Ans: 0,1,2,3,4,5,6,8,∞. The generic solution has 8 circles as drawn below, known as the Circles of Apollonius, ca 200 BC.



https://commons.wikimedia.org/wiki/File:Apollonius8ColorMultiplyV2.svg

Hilbert's 15th Problem

Mathematical Problems by Professor David Hilbert.

Lecture delivered at the ICM, 1900. (Bull.AMS)

15. RIGOROUS FOUNDATION OF SCHUBERT'S ENUMERATIVE CALCULUS.

The problem consists in this: *To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert† especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.*

Although the algebra of to-day guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen.

Schubert's Enumerative Calculus

Hermann Cäsar Hannibal Schubert (22 May 1848 – 20 July 1911)

A German mathematician interested in Enumerative AG.
He wanted a method for finding the typical number of subspaces meeting other given sequences of vector spaces in a given position.

Classical Problem. How many lines meet 4 given lines in $\mathbb{R}^3$?
Ans: 0,1,2,∞.

Modern Problem. How many points, lines, planes,...intersect a given family of Schubert varieties in a fixed set of dimensions?

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(Schubert varieties were named by Bert Kostant, roughly 1960.)

Inspired Consequences

Schubert's calculus and Hilbert's 15th problem inspired many developments in singular homology, cohomoloogy, de Rham cohomology, Chow cohomology, equivariant cohomology, quantum cohomology, intersection theory, cobordism, combinatorics, representation theory and beyond over the past 150 years.

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"by Borel, Marlin, Billey-Haiman and Duan-Zhao, et al. "

Goal. This talk is a Revisionist History of the problem, the solution, and what it continues to inspire.

The Flag Manifold

Defn. A *complete flag* $F_\bullet = (F_1, \ldots, F_n)$ in $\mathbb{C}^n$ is a nested sequence of vector spaces such that $\dim(F_i) = i$ for $1 \le i \le n$. $F_\bullet$ is determined by an ordered basis $\langle f_1, f_2, \ldots f_n \rangle$ where $F_i = \mathrm{span}\langle f_1, \ldots, f_i \rangle$.

Drawn projectively, a flag is a point, on a line, in a plane, ...

Go Schubert Team!

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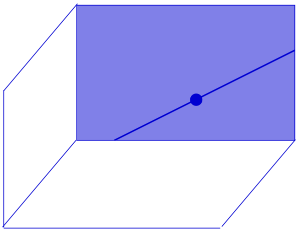
Example $n = 4$ **.**

$$F_\bullet = \langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4, \quad e_2 \rangle$$



The Flag Manifold

Canonical Matrix Form.

$$F_\bullet = \langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4, \quad e_2 \rangle$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2, \quad 2e_1 + e_3, \quad 7e_1 + e_4, \quad e_1 \rangle$$

$\mathcal{F}_n(\mathbb{C}) :=$ *flag manifold* over $\mathbb{C}^n \subset \prod_{k=1}^n Gr(n,k) \subset \prod \mathbb{P}^{\binom{n}{k}}$

$= \{\text{complete flags } F_\bullet\}$

$\approx B \backslash GL_n(\mathbb{C}), \quad B = \text{lower triangular mats.}$

Flags and Permutations

Example.

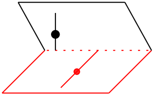
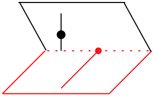
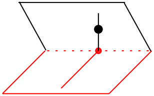
$$F_\bullet = \langle 2e_1 + e_2, \;\; 2e_1 + e_3, \;\; 7e_1 + e_4, \;\; e_1 \rangle \approx \begin{bmatrix} 2 & \textcircled{1} & 0 & 0 \\ 2 & 0 & \textcircled{1} & 0 \\ 7 & 0 & 0 & \textcircled{1} \\ \textcircled{1} & 0 & 0 & 0 \end{bmatrix}$$

Note. If a flag is written in canonical form, the positions of the leading 1's form a permutation matrix. There are 0's to the right and below each leading 1. This permutation determines the *position* of the flag $F_\bullet$ with respect to the reference flag $R_\bullet = \langle e_1, \; e_2, \; e_3, \; e_4 \; \rangle$.

The Flag Manifold

Defn. Consider two complete flags $B_\bullet$ (black) and $R_\bullet$ (red).
Define $\mathrm{pos}(B_\bullet, R_\bullet) := w \in S_n$ if $\dim(B_i \cap R_j) = \mathrm{rk}_{NW}(w[i,j])$

Examples.

	R_1	R_2	R_3	R_4
B_1				1
B_2			1	2
B_3		1	2	3
B_4	1	2	3	4

			1
		1	2
1	1	2	3
1	2	3	4

			1
1	1	1	2
1	1	2	3
1	2	3	4

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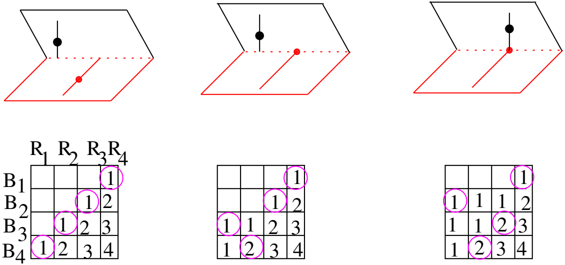


$w = 4321$ $w = 4312$ $w = 4132$

Schubert Cells

Schubert cell $C_w(R_\bullet)$. All flags $F_\bullet$ with $\mathrm{pos}(F_\bullet, R_\bullet) := w$.

$$C_w(R_\bullet) = \{F_\bullet \in \mathcal{F}_n \mid \dim(F_i \cap R_j) = \mathrm{rk}_{NW}(w[i,j])\}$$

Example. $C_{4132}(R_\bullet) = \left\{ \begin{bmatrix} * & * & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} : * \in \mathbb{C} \right\}$

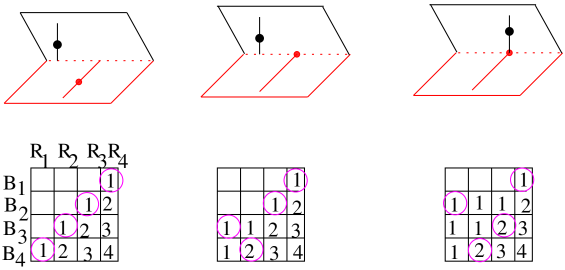
Fact. $\mathtt{dim}_{\mathbb{C}}(C_w) = inv(w) = \#\{(i<j) : w_i > w_j\}$ where $\{(i<j) : w_i > w_j\}$ are the *inversions* of w.

Schubert Varieties

Schubert variety $X_w(R_\bullet)$. is the closure of $C_w(R_\bullet)$.

$$X_w(R_\bullet) = \overline{C_w(R_\bullet)} = \{F_\bullet \in \mathcal{F}_n \mid \dim(F_i \cap R_j) \geq \mathrm{rk}_{NW}(w[i,j])\}$$

Examples.



$$X_{4321} \supset X_{4312} \supset X_{4132}.$$

Fun Facts

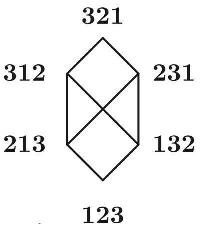
Fact 1. The closure relation on Schubert varieties defines a nice partial order.

$$X_w = \bigcup_{v \le w} C_v \qquad = \bigcup_{v \le w} X_v$$

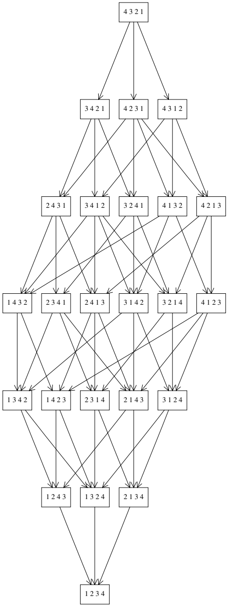
Bruhat order (Ehresmann 1934, Chevalley 1958) is the transitive closure of

$$w < wt_{ij} \iff w(i) < w(j).$$

Example. Bruhat order on permutations in S_3.



Bruhat order on S_4.



10 Fantastic Facts on Bruhat Order

1. Bruhat Order Characterizes Inclusions of Schubert Varieties
2. Contains Young's Lattice in S_∞
3. Nicest Possible Möbius Function
4. Beautiful Rank Generating Functions
5. $[x, y]$ Determines the Composition Series for Verma Modules
6. Symmetric Interval $[\hat{0}, w] \implies X(w)$ rationally smooth
7. Order Complex of (u, v) is shellable
8. Rank Symmetric, Rank Unimodal and k-Sperner
9. Efficient Methods for Comparison
10. Amenable to Pattern Avoidance

Fun Facts

Fact 2. Each Schubert variety $X_w(R_\bullet)$ is an irreducible complex projective variety. As a variety, it is defined as the subset of GL_n/B with certain vanishing determinantal minors based on the rank conditions.

Example. $X_{4132}(R_\bullet) = \overline{\left\{ \begin{bmatrix} * & * & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} : * \in \mathbb{C} \right\}}$

Equations = All 2×2 minors on rows $\{1, 2\}$ and columns $\{2, 3, 4\}$

Schubert Problems

Fix d permutations and d reference flags. The intersection of the corresponding Schubert varieties is a variety

$$Y = X_{w^1}(R^1_\bullet) \cap X_{w^2}(R^2_\bullet) \cap \cdots \cap X_{w^d}(R^d_\bullet).$$

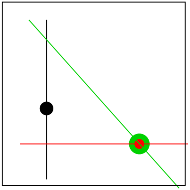
If the reference flags are not in a special position and $Y \neq \emptyset$, then

$$\operatorname{codim}(Y) = \sum_{i=1}^{d} \operatorname{codim}(X_{w^i}) = \sum_{i=1}^{d} \binom{n}{2} - \operatorname{inv}(w^i)$$

Schubert Problem. How many flags are there in the intersection of d Schubert varieties with respect to reference flags in general position if the intersection is 0-dimensional?

Intersecting Schubert Varieties

Example. Fix three flags $R_\bullet$, $G_\bullet$, and $B_\bullet$:



Find all (purple) flags in $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

	R_1	R_2	R_3
P_1		1	
P_2			1
P_3	1		

	G_1	G_2	G_3
P_1			1
P_2		1	
P_3	1		

	B_1	B_2	B_3
P_1		1	
P_2	1		
P_3			1

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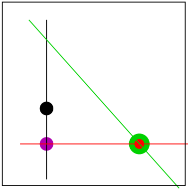

Find all (purple) flags in $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

	R_1	R_2	R_3
P_1		1	
P_2			1
P_3	1		

	G_1	G_2	G_3
P_1			1
P_2		1	
P_3	1		

	B_1	B_2	B_3
P_1		1	
P_2	1		
P_3			1

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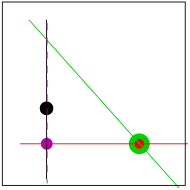


Answer $\{P_\bullet\} = X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

	R_1	R_2	R_3	G_1	G_2	G_3	B_1	B_2	B_3
P_1		1				1		1	
P_2			1		1		1		
P_3	1			1					1

Why is it hard to solve a Schubert Problem?

Schubert Problem. How many flags are there usually in the intersection of d given Schubert varieties if the intersection is 0-dimensional?

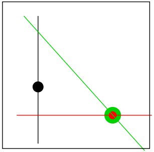
- Solving approx. n^d equations with $\binom{n}{2}$ variables is computationally challenging!
- It's hard guarantee a choice of reference flags is generic.

Observation. We need more information on spans and intersections of flag components, e.g. $\dim(R^1_{x_1} \cap R^2_{x_2} \cap \cdots \cap R^d_{x_d})$.

Permutation Arrays

Theorem. (Eriksson-Linusson, 2000) For every set of d flags $E^1_\bullet, E^2_\bullet, \dots, E^d_\bullet$, there exists a unique permutation array $P \subset [n]^d$ such that

$$\dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d}) = \mathrm{rk}\, P[x].$$



G_1:

	R_1	R_2	R_3
B_1			
B_2			
B_3	①	1	1

G_2:

	R_1	R_2	R_3
B_1			
B_2			①
B_3	1	1	2

G_3:

	R_1	R_2	R_3
B_1			①
B_2		①	2
B_3	1	2	3

Schubert Problems

Def. For a permutation array $P \in [n]^{d+1}$, let

$$C_P = \{F_\bullet \mid \dim(R^1_{x_1} \cap R^2_{x_2} \cap \cdots \cap R^d_{x_d} \cap F_{x_{d+1}}) = \mathrm{rk} P[x]\}.$$

Corollary to EL Theorem. $X = \bigcup_{P \in S(X)} C_P.$

Question. Which P appear if the intersection is 0-dimensional?

Unique Permutation Array Theorem

Theorem.(Billey-Vakil, 2007) Let

$$Y = X_{w^1}(R^1_\bullet) \cap X_{w^2}(R^2_\bullet) \cap \cdots \cap X_{w^d}(R^d_\bullet)$$

be a nonempty intersection of Schubert varieties for generically chosen flags $R^1_\bullet, R^2_\bullet, \ldots, R^d_\bullet$. Then, there exists a unique permutation array $P \in [n]^{d+1}$ such that

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be a nonempty intersection of Schubert varieties for generically chosen flags $R^1_\bullet, R^2_\bullet, \ldots, R^d_\bullet$. Then, there exists a unique permutation array $P \in [n]^{d+1}$ such that

$$Y = C_P.$$

Furthermore, P is completely determined by the d permutations $w^{(1)}, \ldots, s^{(d)}$ via an explicit algorithm, and defining equations for Y can be determined from P.

Realizability Conjecture

Recall. $C_P = \{F_\bullet \mid \dim(E^1_{x_1} \cap \cdots \cap E^d_{x_d} \cap F_{x_{d+1}}) = \mathrm{rk} P[x]\}$

Question. (Eriksson-Linusson) Given any permutation array P, is C_P always nonempty?

- For $d = 2$, yes. Every Schubert variety is nonempty.
- For $d = 3$, yes. Shown by Shapiro-Shapiro-Vainshtein.

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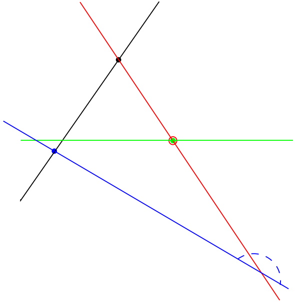
- For $d = 3$, yes. Shown by Shapiro-Shapiro-Vainshtein.
- For $d = 4$, no.

Counterexample



	3		

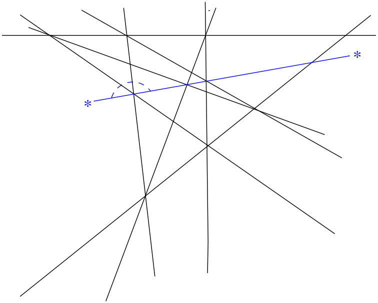
			3.
			$3'$
	3		1

			3
		3	2
	3	2	1

			1
3			
		2	
	0		

- 3. gives realizable array, $3'$ gives an unrealizable array.

Pappus's Hexagon Theorem



- Line $*$ can't hop over the intersection.
- Both ways give valid permutation arrays in $[3]^9$.

Conclusions/Questions

- We give equations for all 0-dimensional Schubert problems using permutation arrays.
- Question: How can these equations most efficiently be solved?

- We have shown that not every permutation array is realizable.
- Hard Question: Can we identify all realizable permutation arrays?(probably not)

- Sometimes we can determine $c_{uv}^{w} = 0$ by looking just at the corresponding permutation array P.
- Question:Can we identify all zero coefficients this way? (see Purbhoo 2006, St. Dizier-Yong 2020)