Boolean product polynomials, Schur positivity, and Chern plethysm

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Based on joint work with:
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Garsiafest June 19, 2019

Hope, Determination, Passion for Math and for Life

Left to right: Theresa Gallo, Ethan Reiner, Hélène Barcelo, Luisa Carini, Ed Allen, S.B., Joaquin Carbonara, Adriano Garsia, Alain Goupil, Ezra Halleck

Hope, Determination, Passion for Math and for Life

Mark Haiman, Greg Warrington, S.B., Brendan Pawlowski, Adriano Garsia, Josh Swanson, Sean Griffin

. . . and Passion for Good Food

Adriano at Pike Place Market in preparation for cooking the Fish Couscous.

Please use your cookbook to prepare a meal in Adriano's style. Use lots of salt, and everything is cooked at 400 degrees! Enjoy!

Outline

Symmetric Polynomials

Boolean Product Polynomials

Chern Plethysm

Schur Positivity via GL_n representation theory and vector bundles

Open Problems

Symmetric Polynomials

Notation.

- Fix an alphabet of variables $X = \{x_1, x_2, \dots, x_n\}$.
- The symmetric group S_n acts on $\mathbb{C}[x_1, x_2, \dots, x_n]$ by permuting the variables: $w.x_i = x_{w(i)}$.
- A polynomial $f \in \mathbb{C}[x_1, x_2, \dots, x_n]$ is *symmetric* if $w.f = f$ for all $w \in S_n$.
- Let Λ_n denote the *ring of symmetric polynomials* in $\mathbb{C}[x_1, x_2, \dots, x_n]$.

Symmetric Polynomials

Examples. Let $[n] = \{1, 2, \ldots, n\}$.

Elementary: $$e_k = \sum_{\substack{A \subset [n] \\ |A| = k}} \prod_{i \in A} x_i$$

Homogeneous: $$h_k = \sum_{\substack{\text{multisets } A \subset [n] \\ |A| = k}} \prod_{i \in A} x_i$$

Power sum: $$p_k = \sum_{i=1}^{n} x_i^k$$

$$e_2(x_1, x_2, x_3, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

$$p_2(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

$$h_2 = e_2 + p_2.$$

Symmetric Polynomials

Fact. $\Lambda_n = \mathbb{C}[e_1, \ldots, e_n] = \mathbb{C}[h_1, \ldots, h_n] = \mathbb{C}[p_1, \ldots, p_n]$

Symmetric Polynomials

Fact. $\Lambda_n = \mathbb{C}[e_1, \ldots, e_n] = \mathbb{C}[h_1, \ldots, h_n] = \mathbb{C}[p_1, \ldots, p_n]$

Question. What other symmetric polynomials are "natural"?

Symmetric Polynomials

Fact. $\Lambda_n = \mathbb{C}[e_1, \ldots, e_n] = \mathbb{C}[h_1, \ldots, h_n] = \mathbb{C}[p_1, \ldots, p_n]$

Question. What other symmetric polynomials are "natural"?

Monomials: $m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} +$ other monomials in S_n-orbit

Stanley's chromatic symmetric functions on a graph $G = (V, E)$:

$$X_G(x_1, \ldots, x_n) = \sum_{\substack{c : V \to [n] \\ \text{proper coloring}}} \prod_{v \in V} x_{c(v)}.$$

Observe. These examples are all sums of products.

Schur Polynomials

Defn. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, the *Schur polynomial*

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in SSYT(\lambda, n)} \prod_{i \in T} x_i$$

where $SSYT(\lambda, n)$ are the semistandard fillings of λ with positive integers in $[n]$. Semistandard implies strictly increasing in columns and leniently increasing in rows.

Example. For $\lambda = (2, 1)$ and $n = 2$, $SSYT(\lambda, n)$ has two fillings

2	
1	1

2	
1	2

so $s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$.

Boolean Product Polynomials

Question. What about products of sums?

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Defn. For $X = \{x_1, \ldots, x_n\}$, define

- *(n,k)-Boolean Product Polynomial*: For $1 \leq k \leq n$,

$$B_{n,k}(X) := \prod_{\substack{A \subseteq [n] \\ |A| = k}} \sum_{i \in A} x_i$$

- *n-th Total Boolean Product Polynomial*:

$$B_n(X) := \prod_{k=1}^{n} B_{n,k}(X) = \prod_{\substack{A \subseteq [n] \\ A \neq \emptyset}} \sum_{i \in A} x_i$$

Example. $B_2 = (x_1)(x_2)(x_1 + x_2) = x_1^2 x_2 + x_1 x_2^2 = s_{(2,1)}(x_1, x_2)$

Boolean Product Polynomials

Examples.

$$B_{3,1} = (x_1)(x_2)(x_3) = e_3(x_1, x_2, x_3) = s_{(1,1,1)}(x_1, x_2, x_3)$$

$$B_{3,2} = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) = s_{(2,1)}(x_1, x_2, x_3)$$

$$B_{3,3} = (x_1 + x_2 + x_3) = e_1(x_1, x_2, x_3) = s_{(1)}(x_1, x_2, x_3)$$

$$B_3 = s_{(1,1,1)} s_{(2,1)} s_{(1)} = s_{(4,2,1)} + s_{(3,3,1)} + s_{(3,2,2)}.$$

Motivation from Lou Billera (BIRS 2015)

Defn. Consider the real variety $V(B_n)$. Since each factor of B_n is linear, this variety is a hyperplane arrangement called the *Resonance Arrangement* or *All-Subsets Arrangement* $\mathcal{H}_n$. Each hyperplane is orthogonal to a nonzero 0-1-vector in $\mathbb{R}^n$.

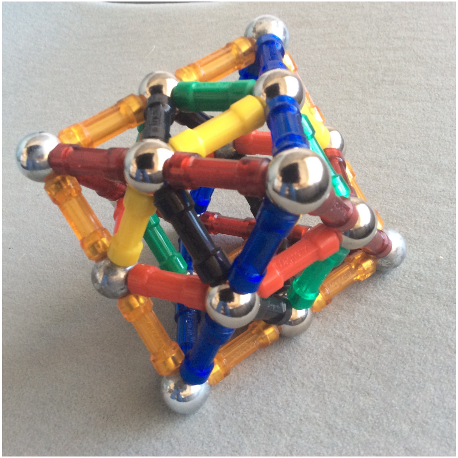

Photo by Lou Billera

Motivation

1. The regions of $\mathcal{H}_n$ are the domains of polynomiality of double Hurwitz numbers. See Cavalieri-Johnson-Markwig, 2011.

2. The chambers of the Resonance Arrangement $\mathcal{H}_n$ can be labeled by maximal unbalanced collections of 0-1 vectors. See Billera-Moore-Moraites-Wang-Williams, 2012.

3. Minimal balanced collections determine the minimum linear description of cooperative games possessing a nonempty core in Lloyd Shapley's economic game theory work from 1967. Finding a good formula for enumerating them is still open.

4. The order complex for the "Positive Set Sums" is closely related to the regions of the resonance arrangement and unbalanced collections of 0-1 vectors. See Björner, 2015.

Further Connections. See Lou Billera's talk slides *"On the real linear algebra of vectors of zeros and ones"*

Subset Alphabets

Defn. For $1 \le k \le n$, define a new alphabet of linear forms

$$X^{(k)} = \{x_A = \sum_{i \in A} x_i : A \subset [n], |A| = k\}.$$

Then

$$B_{n,k} = \prod_{\substack{A \subseteq [n] \\ |A| = k}} x_A = e_{\binom{n}{k}}(X^{(k)}).$$

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Furthermore, for $1 \leq p \leq \binom{n}{k}$ define the symmetric polynomials

$$e_p(X^{(k)}) = \sum_{\substack{S \subset k\text{-subsets of}[n] \\ |S| = p}} \prod_{A \in S} x_A.$$

Schur Positivity

Theorem. For all $1 \leq k \leq n$ and $1 \leq p \leq \binom{n}{k}$, the expansion

$$e_p(X^{(k)}) = \sum_{\lambda} c_\lambda s_\lambda(x_1, \ldots, x_n)$$

has nonnegative integer coefficients c_λ.

Corollary. Both $B_{n,k}$ and B_n are Schur positive.

Proof Setup

Notation. Fix a complex vector bundle $\mathcal{E}$ of rank n over a smooth complex projective variety X. The *total Chern class* $c(\mathcal{E})$ is the sum of the individual Chern classes

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \cdots + c_n(\mathcal{E}) \in H^\bullet(X, \mathbb{Z}).$$

Via the Splitting Principle, there exist elements $x_1, x_2, \ldots, x_n$ in $H^2(X)$, unique up to permutation, such that

$$c(\mathcal{E}) = \prod_{i=1}^{n} (1 + x_i).$$

The x_i's are the *Chern roots* of $\mathcal{E}$ associated to certain line bundles.

Chern Plethysm

Defn. Given a complex rank n vector bundle $\mathcal{E} \twoheadrightarrow X$ with Chern roots, $x_1, x_2, \ldots, x_n$, and a symmetric function $F \in \Lambda_\infty$, define the *Chern Plethysm* $F(\mathcal{E}) = F(x_1, \ldots, x_n, 0, 0, 0, \ldots) \in H^\bullet(X)$.

If F is homogeneous of degree d, then $F(\mathcal{E})$ is a polynomial of degree d so $F(\mathcal{E}) \in H^{2d}(X)$.

Chern Plethysm

Key Fact. Operations on vector spaces induce operations on vector bundles, which induce **linear** changes in Chern roots.

Example 1.

- $\mathcal{E} \twoheadrightarrow X$ be a vector bundle with Chern roots $x_1, \ldots, x_n$,
- $\mathcal{F} \twoheadrightarrow X$ be a vector bundle with Chern roots $y_1, \ldots, y_m$.

Whitney sum formula: the Chern roots of the direct sum $\mathcal{E} \oplus \mathcal{F}$ are the multiset union of $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ so that

$$F(\mathcal{E} \oplus \mathcal{F}) = F(x_1, \ldots, x_n, y_1, \ldots, y_m).$$

Since F is symmetric, the ordering of these entries doesn't matter.

Chern Plethysm

Key Fact. Operations on vector spaces induce operations on vector bundles which in turn induce linear changes in the Chern roots.

Example 2. For the tensor product bundle $\mathcal{E} \otimes \mathcal{F}$, the Chern roots are the multiset of sums $x_i + y_j$ where $1 \leq i \leq n$ and $1 \leq j \leq m$ so that

$$F(\mathcal{E} \otimes \mathcal{F}) = F(\overbrace{\ldots, x_i + y_j, \ldots}^{1 \leq i \leq n,\ 1 \leq j \leq m}).$$

Chern Plethysm

Example 3. Let λ be a partition, and let $\mathbb{S}^\lambda$ denote the Schur functor. Apply $\mathbb{S}^\lambda$ to the bundle $\mathcal{E}$ to obtain a new bundle $\mathbb{S}^\lambda(\mathcal{E})$ with fibers $\mathbb{S}^\lambda(\mathcal{E})_p := \mathbb{S}^\lambda(\mathcal{E}_p)$.

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Thm. (Fulton) The Chern roots of $\mathbb{S}^\lambda(\mathcal{E})$ are the multiset of sums $\sum_{c\in\lambda} x_{T(c)}$ where T varies over $\mathrm{SSYT}(\lambda, \leq n)$ so that

$$F(\mathbb{S}^\lambda(\mathcal{E})) = F(\overbrace{\ldots, \sum_{c\in\lambda} x_{T(c)}, \ldots}^{T\in\mathrm{SSYT}(\lambda,\leq n)})$$

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$$F(\mathbb{S}^\lambda(\mathcal{E})) = F(\overbrace{\dots, \sum_{c\in\lambda} x_{T(c)}, \dots}^{T\in\mathrm{SSYT}(\lambda,\leq n)})$$

For example, if $\lambda = (2,1)$ and $n = 3$, the $\mathrm{SSYT}(\lambda, \leq n)$ are

$$\begin{array}{|c|}\hline 2\\ \hline\end{array}\!\!\begin{array}{l}\\ \begin{array}{|c|c|}\hline 1&1\\ \hline\end{array}\end{array}$$

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{1}\\ \hline\end{array}\quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{2}\\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{1}\\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2}\\ \hline\end{array}\quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{3}\\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{|c|}{2}\\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{3}\\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{|c|}{2}\\ \hline\end{array}$$

$$F(\mathbb{S}^\lambda(\mathcal{E})) = F(2x_1+x_2, x_1+2x_2, 2x_1+x_3, x_1+x_2+x_3, x_1+x_2+x_3,$$

$$2x_2+x_3, x_1+2x_3, 2x_2+x_3)$$

Laws of Evaluation for Chern Plethysm

If $\mathcal{E}$ is any vector bundle, $F, G \in \Lambda_\infty$ are any symmetric functions and $\alpha, \beta \in \mathbb{C}$ are scalars, we have

$$\begin{cases} (F \cdot G)(\mathcal{E}) = F(\mathcal{E}) \cdot G(\mathcal{E}), \\ (\alpha F + \beta G)(\mathcal{E}) = \alpha F(\mathcal{E}) + \beta G(\mathcal{E}), \\ \alpha(\mathcal{E}) = \alpha. \end{cases}$$

Laws of Evaluation for Classical Plethysm

Defn. Let $E = E(t_1, t_2, \dots)$ be any rational function, and let F be a symmetric function. The *classical plethysm $F[E]$* is determined by the relations

$$\begin{cases} (F \cdot G)[E] = F[E] \cdot G[E], \\ (\alpha F + \beta G)[E] = \alpha F[E] + \beta G[E], \\ \alpha[E] = \alpha \end{cases}$$

for all symmetric functions $F, G \in \Lambda_\infty$ and scalars $\alpha, \beta \in \mathbb{C}$, along with

$$p_k[E] = p_k[E(t_1, t_2, \dots)] := E(t_1^k, t_2^k, \dots), \quad k \geq 1.$$

Since $\Lambda_\infty = \mathbb{C}[p_1, p_2, \dots]$ this defines $F[E]$ uniquely.

Chern Plethysm vs Classical Plethysm

Example. If $E = s_{(2,1)}(x_1, x_2, x_3)$

$$\begin{array}{ll}2&\\1&1\end{array}\quad\begin{array}{ll}2&\\1&2\end{array}\quad\begin{array}{ll}3&\\1&1\end{array}\quad\begin{array}{ll}3&\\1&2\end{array}\quad\begin{array}{ll}2&\\1&3\end{array}\quad\begin{array}{ll}3&\\2&2\end{array}\quad\begin{array}{ll}3&\\1&3\end{array}\quad\begin{array}{ll}3&\\2&2\end{array}$$

$$F[E] = F(x_1^2x_2, x_1x_2^2, x_1^2x_3, x_1x_2x_3, x_1x_2x_3, x_2^2x_3, x_1x_3^2, x_2^2x_3)$$

Key Difference.

- **Classical Plethysm**: If E is a polynomial of degree e, the degree of $F[E]$ is $e \cdot \deg(F)$.
- **Chern Plethysm** For any bundle $\mathcal{E}$, the degree of $F(\mathcal{E})$ equals the degree $\deg(F)$ of F.

Recall Subset Alphabets and Boolean Product Polynomials

Defn. For $1 \leq k \leq n$, recall the *alphabet of linear forms*

$$X^{(k)} = \{x_A = \sum_{i \in A} x_i : A \subset [n], |A| = k\}.$$

Then

$$B_{n,k} = \prod_{\substack{A \subseteq [n] \\ |A| = k}} x_A = e_{\binom{n}{k}}(X^{(k)}).$$

Goal. Prove Schur positivity for all $1 \leq p \leq \binom{n}{k}$ of

$$e_p(X^{(k)}) \; = \; \sum_{\substack{S \subset k\text{-subsets of} [n] \\ |S| = p}} \; \prod_{A \in S} x_A \; = \; e_p(\wedge^k \mathcal{E}).$$

Prior Work

Thm.(Lascoux, 1978) The total Chern class of $\bigwedge^2\mathcal{E}$ and $\mathrm{Sym}^2\mathcal{E}$ is Schur-positive in terms of the Chern roots $x_1, \ldots, x_n$ of $\mathcal{E}$. Specifically, there exist integers $d_{\lambda,\mu} \geq 0$ for $\mu \subseteq \lambda$ such that

$$c(\textstyle\bigwedge^2\mathcal{E}) = \prod_{1\leq i<j\leq n}(1+x_i+x_j) = 2^{-\binom{n}{2}}\sum_{\mu\subseteq\gamma_n} d_{\gamma_n,\mu}2^{|\mu|}s_\mu(X),$$

$$c(\mathrm{Sym}^2\mathcal{E}) = \prod_{1\leq i\leq j\leq n}(1+x_i+x_j) = 2^{-\binom{n}{2}}\sum_{\mu\subseteq\delta_n} d_{\delta_n,\mu}2^{|\mu|}s_\mu(X).$$

Here $\gamma_n = (n-1, \ldots, 1, 0)$ and $\delta_n = (n, \ldots, 2, 1)$.

Binomial Determinants

Lascoux showed that for $\mu = (\mu_1, \ldots, \mu_n) \subseteq \lambda = (\lambda_1, \ldots, \lambda_n)$,

$$d_{\lambda,\mu} = \det\left(\binom{\lambda_i + n - i}{\mu_j + n - j}\right)_{1\le i,j\le n} \ge 0.$$

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Thm.(Gessel-Viennot 1985) $d_{\lambda,\mu}$ counts the number of nonintersecting lattice paths from heights $\lambda + \delta_n$ along the y-axis to main diagonal points $\mu + \delta_n$ using east or south steps.

This highly influential theorem was inspired by Lascoux's theorem!

Key Ingredient

Thm.(Pragacz 1996) Let λ be a partition, and let

- $\mathcal{E}_1, \ldots, \mathcal{E}_k$ be vector bundles,
- $Y_1, \ldots, Y_k$ be the alphabets consisting of their Chern roots,
- $\mu^{(1)}, \ldots, \mu^{(k)}$ be partitions.

Then, there exists nonnegative integers $c_{(\nu^{(1)},\ldots,\nu^{(k)})}$ such that

$$s_\lambda(\mathbb{S}^{\mu^{(1)}}(\mathcal{E}_1)\otimes\cdots\otimes\mathbb{S}^{\mu^{(k)}}(\mathcal{E}_k)) = \sum_{\nu_1,\ldots,\nu_k} c_{(\nu^{(1)},\ldots,\nu^{(k)})} s_{\nu_1}(Y_1)\cdots s_{\nu_k}(Y_k).$$

Pragacz's proof uses work of Fulton-Lazarsfeld on numerical positivity for ample vector bundles. The Hard Lefschetz theorem is a key component.

Corollaries

Cor. For any partitions λ, μ, the Chern plethysm $s_\mu(\mathbb{S}^\lambda(\mathcal{E}))$ is Schur positive.

Cor. The expansion of $e_p(X^{(k)})$ is Schur positive since the Chern roots of $\mathbb{S}^{1^k}(\mathcal{E}) = X^{(k)}$ and $e_p = s_{1^p}$, including the Boolean product polynomials.

Cor. The analogue of Lascoux's theorem holds for all Schur functors $\mathbb{S}^\lambda(\mathcal{E}))$, e.g.

$$c(\bigwedge^k \mathcal{E}) = \prod_{A \subseteq [n], |A|=k} \left(1 + \sum_{i \in A} x_i\right) = \sum_{p \geq 0} e_p(X^{(k)}).$$

Question. What are the Schur expansions?

Boolean Product Expansions for $B_{n,n-1}$

Thm.(Désarménien-Wachs + Reiner-Webb)

$$B_{n,n-1} = \prod_{i=1}^{n}(x_1 + x_2 + \ldots + x_n - x_i) = \sum_{\lambda \vdash n} a_\lambda s_\lambda(X)$$

where a_λ is the number of $T \in SYT(\lambda)$ with smallest ascent given by an even number, where n is always considered to be an ascent.

Thm.(Gessel-Reutenauer)

$$B_{n,n-1} = \sum_{w \in D_n} F_{D(w)} = \sum_{\lambda} \mathrm{ch}(\mathrm{Lie}_\lambda(\mathbb{C}^n)).$$

where sum ranges over all partitions $\lambda \vdash n$ which have no parts of size 1 and $\mathrm{Lie}_\lambda(\mathbb{C}^n)$ is a GL_n-representation called a *higher Lie module*.

Boolean Product Expansions for $B_{n,n-1}$

More generally, consider

$$B_{n,n-1}(X;q) := \prod_{i=1}^{n}(x_1 + x_2 + \ldots + x_n + qx_i)$$
$$= \sum_{j=0}^{n} q^j e_j(X) h_{(1^{n-j})}(X).$$

Question. Is $B_{n,n-1}(X;q)$ as the graded Frobenius characteristic of an interesting module?

S_n module for $B_{n,n-1}$

Building on the recent success of Haglund-Rhoades-Shimozono and Pawlowski-Rhoades, . . .

Defn. For $n \geq 0$, the *Superspace Algebra* is the associative unital $\mathbb{C}$-algebra with generators $x_1, \ldots, x_n, \theta_1, \ldots, \theta_n$ subject to the relations for all $1 \leq i, j \leq n$,

$$x_i x_j = x_j x_i, \quad x_i \theta_j = \theta_j x_i, \quad \theta_i \theta_j = -\theta_j \theta_i.$$

Let $\mathbb{C}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]$ denote superspace, with the understanding that the x-variables commute and the θ-variables anticommute.

The symmetric group $\mathfrak{S}_n$ acts on superspace diagonally by the rule

$$w.x_i := x_{w(i)}, \quad w.\theta_i := \theta_{w(i)}, \quad w \in \mathfrak{S}_n, \ 1 \leq i \leq n.$$

A Quotient of Superspace

Defn. The *divergence free quotient of superspace* is defined by

$$DF_n := \mathbb{C}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n] / \langle x_1\theta_1, x_2\theta_2, \ldots, x_n\theta_n \rangle.$$

Observe, DF_n is a bigraded S_n module since the generators of the defining ideal are homogeneous and invariant under the S_n action on superspace.

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Thm.(Billey-Rhoades-Tewari)

$$\mathrm{grFrob}(DF_n; q, t) = \sum_{j=0}^{n} q^j \cdot e_j(X) \cdot h_{n-j}\left[\frac{X}{1-t}\right].$$

A Second Quotient of Superspace

Defn. Let $I_n = \langle e_1(X_n), e_2(X_n), \ldots, e_n(X_n) \rangle \subseteq DF_n$ be the ideal generated by the n elementary symmetric polynomials in the x-variables.

Let $R_n := DF_n / I_n$ be the bigraded quotient $\mathfrak{S}_n$-module.

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Thm.(Billey-Rhoades-Tewari)

$$\mathrm{grFrob}(R_n; q, t) = \sum_{j=0}^{n} q^j \cdot e_j(X) \cdot \left[\sum_{T \in \mathrm{SYT}(n-j)} t^{\mathrm{maj}(T)} \cdot s_{\mathrm{shape}(T)}(X) \right].$$

Cor. $\mathrm{grFrob}(R_n; q, 1) = \sum_{j=0}^{n} q^j \cdot e_j(X) \cdot h_{(1^{n-j})}(X) = B_{n,n-1}(X; q).$

Lots of Beautiful Open Problems!

1. Find the Schur expansion of the Chern plethysm $s_\lambda(\mathbb{S}^\mu(\mathcal{E}))$.

2. Find the Chern plethysm analog of the classical plethysm $s_\lambda[s_\mu]$ interpretation as the Weyl character of $\mathbb{S}^\lambda(\mathbb{S}^\mu(\mathbb{C}^n))$. One possible approach is to let V be the vector space $\mathrm{Hom}_{\mathbb{C}}(\mathbb{S}^\lambda(\mathbb{S}^\mu(\mathbb{C}^n)), (\mathbb{C}^{|\mu|})^{\otimes|\lambda|})$. Find an action of GL_n on V whose Weyl character equals $s_\lambda(\mathbb{S}^\mu(\mathcal{E}))$.

3. What other interesting polynomials are Schur positive via Chern plethysm?

Many Thanks Adriano and All of YOU!

For sharing your joy of mathematics research, your support throughout my career, your commitment to students and to our research community! You built this House of Math!