Existence and Hardness of Conveyor Belts

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Based on joint work with:
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Outline

Inspiration from Erik and Marty Demaine

What is a conveyor belt?

Existence?

Computational Complexity?

Open Problems?

Demaine's Rules of Conduct for an Open Problem Session

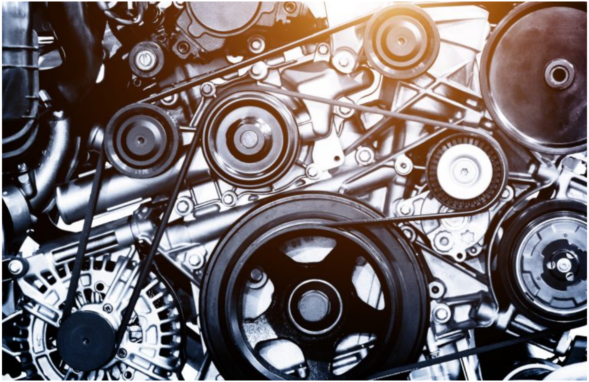
This work was initiated during an open problem session held at the University of Washington in fall 2016 while the Demaines were Walker–Ames Lecturers, sponsored by the UW Graduate School. There were 25 students and faculty in math, art, and computer science who attended the session and tackled the problem.

1. We will not discuss these problems or our progress with anyone before the paper is finished.
2. When we solve this problem and submit it for publication, everyone can say if they deserve to be an author or not based on their own assessment of their contribution.
3. The authors on the paper will appear in alphabetical order by last name.

What is a conveyor belt?

https://www.bostonmagazine.com/restaurants/2016/08/18/yo-sushi-conveyor-belt-boston/

What is a conveyor belt?



https://gobdp.com/blog/engine-belts-squeal/

What is a conveyor belt?

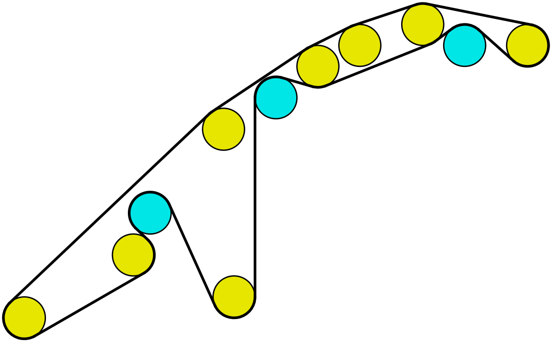
https://dinostire.com/belt-tips/

The Conveyor Belt Problem

In 2001, Manuel Abellanas asks. . .

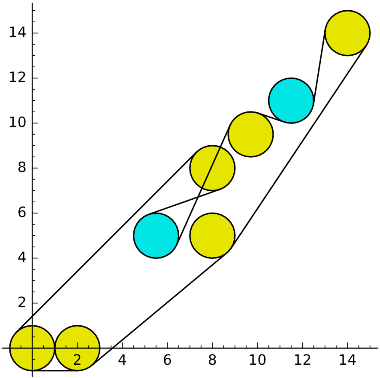
Can every set of disjoint closed disks in the plane be simultaneously touched by a conveyor belt, which means a tight elastic band that touches the boundary of each disk, possibly multiple times?

Example.



Conveyor Belts

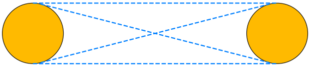
Example of a tight closed curve that is not a legal conveyor belt.



Conveyor Belts

Def. Given a finite collection $\mathcal{D}$ of disjoint closed disks in the plane, a *conveyor belt* is a continuously differentiable simple closed curve that touches the boundary of each disk at least once, is disjoint from the disk interiors, and consists of nonoverlapping arcs of the disks and nonintersecting bitangents between them.

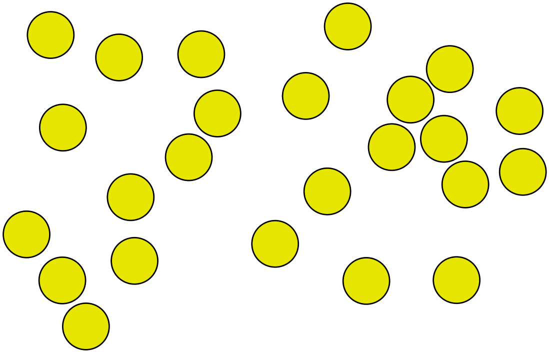
There are 4 bitangents for any two nonoverlapping disks



The two disks will both be on the same side of the band iff it includes the upper or lower bitangents.

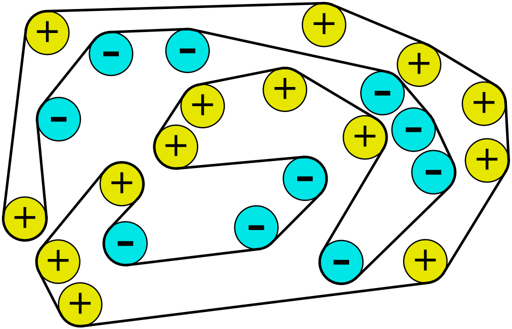
Conveyor Belts

Does this collection of disks have a conveyor belt?



Conveyor Belts

Yes! This one was found via computer assisted exploration aka "machine learning".



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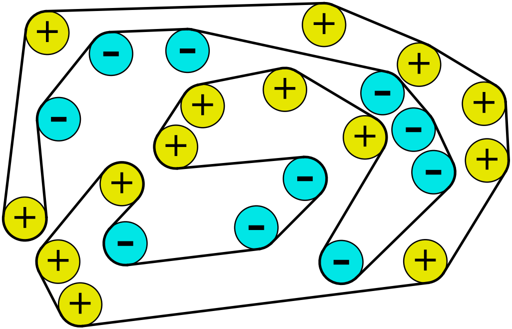

Note, we can encode a conveyor belt by specifying the order of disks touched by the belt up to cyclic rotation and specifying which disks turn in the same direction when the belt is pulled.

Existence?

Question. Does every set of disjoint closed disks in the plane have a conveyor belt wrapping?

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Answer. No!

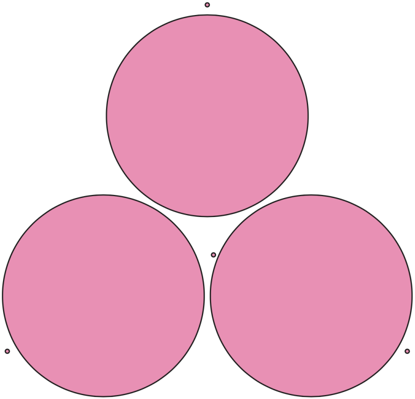

Figure 3: Seven disks of different sizes with no valid wrapping. [Javier Tejel and Alfredo García]

Existence?

Open Question. (Abellanas) Does every set of disjoint closed *unit* disks in the plane have a conveyor belt wrapping?

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Input. Suppose we have a sequence of n disks with centers (x_i, y_i) for $1 \leq i \leq n$. By rotating and relabeling the disks if necessary, we can assume without loss of generality that $x_1 < x_2 < \cdots < x_n$, so the disks are sorted by their x-coordinates.

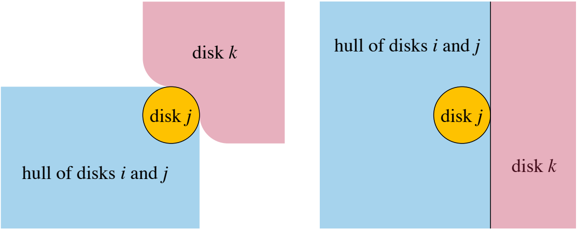
Modified Existence Question, Complexity, and Enumeration

Demaine's Questions.

1. Does every set of disjoint closed *unit* disks in the plane have a conveyor belt wrapping if we require both the x-coordinates and the y-coordinates to be increasing?
2. What is the computational complexity of the problem of determining if a conveyor belt exists for a given configuration of disks of arbitrary radii?
3. What is the minimum and maximum number of distinct conveyor belts of any configuration of n disks? Note, two belts are equivalent if the sequence of disks visited by the belts are related by a cyclic permutation and maintain the inside/outside designations of each disk.

Modified Existence Question

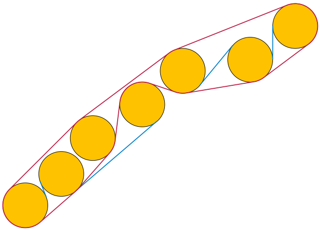
Def. We say that a sequence of disks is *monotonically separated* if $x_1 < x_2 < \cdots < x_n$ and for every $i < j < k$, the kth disk is disjoint from the convex hull of the ith and jth disks, and the ith disk is disjoint from the convex hull of the jth and kth disks.



Modified Existence Question

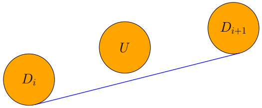
Theorem 1. Every monotonically separated sequence of unit disks has a conveyor belt which can be constructed in linear time, once the disks have been sorted by their x-coordinates.

Proof. Use the convex hull along the top, and then follow a linear time algorithm to wind the belt through all other disks.

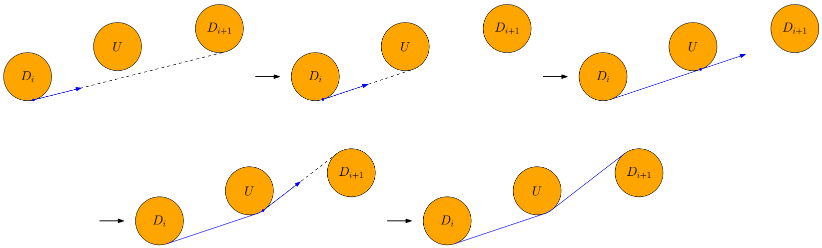


Winding Process

The first partial conveyor belt between D_i and D_{i+1} produced by the winding process, where U is a disk on the upper hull.

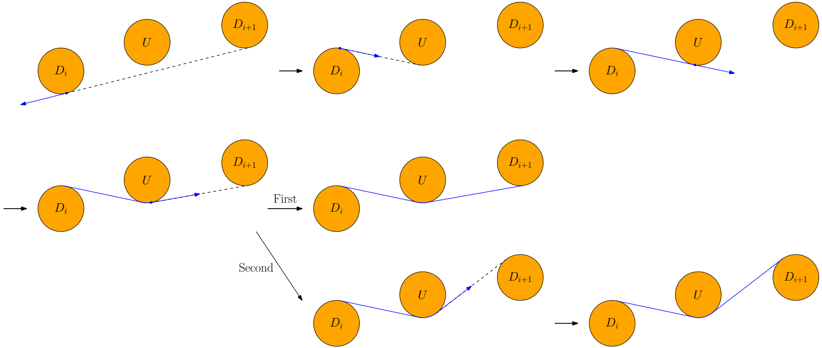


An "animation" of the second part of the winding process, which produces a second partial conveyor belt in the convex hull of D_i and D_{i+1}.



UnWinding Process

Two more partial conveyor belt between D_i and D_{i+1} are produced by the unwinding process.



Nice properties

We claim the winding and unwinding process has the following properties.

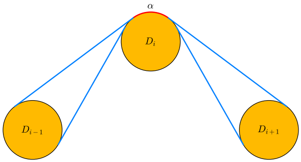
(a) Both partial conveyor belts lie entirely within the convex hull of D_i and D_{i+1}.

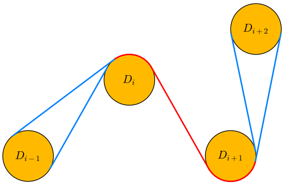
(b) Both belts are disjoint from the upper convex hull, even though some upper hull disks may be touched by either belt.

(c) Both belts are valid partial conveyor belts touching D_i and D_{i+1} in the sense that they consist of arcs of disks and bitangents between them whose union is a continuously differentiable curve without self-intersection that is disjoint from all disk interiors.

Gluing Partial Conveyor Belts

There always exists a way to glue two of these four partial belts to two of the four partial belts for the next pair. Extend until done.





Linear time follows if disks come sorted already because there are at most two upper hull disks touched by a partial belt between D_i and D_{i+1}.

Complexity Question

Def. A yes/no existence problem or decision problem is in *NP = Nondeterministic Polynomial time* if one can verify that a proposed example is an acceptable solution in polynomial time.

Def. A yes/no existence problem is *NP-complete* if there exists a polynomial time algorithm that converts the input to any other NP problem to this problem's input and a polynomial time algorithm to reconstruct the answer to the original problem from a black-box solution to this problem.

For example, determining if a graph has a Hamiltonian cycle is NP-complete, determining if a multiset of integers contains a subset that sums to 0, and winning at Solitaire Battleship.

Complexity Question

Theorem 2.
The one-touch conveyor belt problem is NP-complete.

Theorem 3.
It is NP-complete to determine whether a given system of disks has a conveyor belt, even allowing the belt to touch a single disk multiple times along disjoint arcs.

Complexity Question

Theorem 2. The one-touch conveyor belt problem is NP-complete.

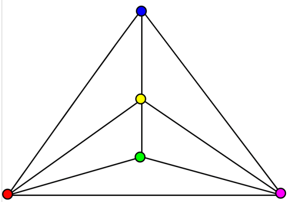
Proof Sketch. Given an ordered lists of disks touched by a belt and a list of which disks are inside the belt, it is polynomial time verifiable to test if the given tight simple closed curve is a conveyor belts. Thus, the Conveyor Belt problem is in NP.

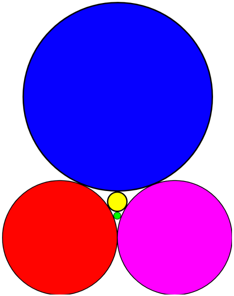
To prove NP-hardness, we reduce from a known NP-complete problem, determining the existence of a Hamiltonian cycle in a maximal planar graph. This problem was proven NP-complete by Wigderson in 1982.

Complexity Question

Proof Sketch cont. Given a maximal planar graph G, we construct a disk placement using the Circle Packing Theorem. We show the the disk placement has a one touch conveyor belt if and only if G has a Hamiltonian cycle.

Circle Packing Theorem. (Koebe,Andreev,Thurston): Given a maximal planar graph G, there exists a packing in the plane whose adjacency graph is isomorphic to G.





https://en.wikipedia.org/wiki/Circle_packing_theorem

Enumeration Question

Still Open Problem. What is the minimum and maximum number of distinct conveyor belts of any configuration of n disks?

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Def. A problem is in #P if it is the counting problem associated to an NP problem. It is #P-complete, if it is #P-hard.

Question. Is counting the number of conveyor belts for a configuration of disks #P-complete?

Related Work

1. Manuel Abellanas. Conectando puntos: poligonizaciones y otros problemas relacionados. Gaceta de la Real Sociedad Matematica Española, 2008.
2. Demaine, Demaine, and Palop (2010) designed puzzle fonts, specified by a system of disks per character, with a unique conveyor belt in the shape of that character. If the conveyor belt is not shown, decoding the font becomes a puzzle for the viewer.
3. Joe O'Rourke (2011) relaxed the problem by allowing the curve to cross itself or wrap around some arcs of disks more than once, with prescribed disk orientations. For his variant of the problem, not every system of disjoint unit disks has a conveyor belt, but for a belt of this type to exist it is sufficient for a certain *hull-visibility graph* of the disks to be connected or for the disks to remain disjoint when expanded by a sufficiently large factor.

Other Open Problems

1. Can one identify a larger class of unit disks, larger than monotonically separated, which has a conveyor belt? Could one use O'Rouke's hull visibility graph to identify such a beltable collection of unit disks?

2. Is the problem of finding the number of conveyor belts for a given disk configuration #P-complete?

3. Given n points in the plane, is finding the number of polygons with vertices at the n points also #P-complete?

Many Thanks for Listening!