A Pattern Avoidance Characterization for Smoothness of Positroid Varieties

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Based on joint work with:
Jordan Weaver and Christian Krattenthaler

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Cascade Lectures in Combinatorics
April 22, 2023 Slides:

math.washington.edu/~billey/talks/reed.pdf

Quote. “Combinatorics is the nanotechnology of mathematics”

Sara Billey, 2005

Outline

Motivation: Pattern avoidance on Permutations

Pattern avoidance on Decorated Permutations

Positroid varieties

Characterizing Smooth Positroid Varieties

Enumeration

Future Work

Permutations

Defn. A *permutation* w in the symmetric group S_n is a bijection from $[n] = \{1, 2, \ldots, n\}$ to itself, denoted in one-line notation $w = [w(1), w(2), \ldots, w(n)]$.

Example. $[4, 2, 1, 5, 6, 3] \in S_6$

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$$[1, 27, 2, 28, 3, 29, 4, 30, 5, 31, 6, 32, 7, 33, 8, 34, 9, 35, 10, 36,$$

$$11, 37, 12, 38, 13, 39, 14, 40, 15, 41, 16, 42, 17, 43, 18, 44, 19, 45,$$

$$20, 46, 21, 47, 22, 48, 23, 49, 24, 50, 25, 51, 26, 52]$$

Pattern Avoidance on Permutations

Defn. A *permutation* $w = [w(1), w(2), \ldots, w(n)] \in S_n$ *contains* another permutation $v = [v(1), v(2), \ldots, v(k)] \in S_k$ for $k \leq n$ provided there exists $1 < i_1 < i_2 < \ldots < i_k \leq n$ such that $[w(i_1), w(i_2), \ldots, w(i_k)]$ and $[v(1), v(2), \ldots, v(k)]$ have the same relative order.

If no length k subsequence of $[w(1), w(2), \ldots, w(n)]$ has the same relative order as $[v(1), v(2), \ldots, v(k)]$ then we say w *avoids* v.

Example. Take $v = [4, 2, 3, 1]$. Then,

$w = [6, 2, 5, 4, 3, 1]$ contains $6241 \sim 4231$

$w = [6, 1, 2, 5, 4, 3]$ avoids 4231

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Question. Does the perfect shuffle permutation avoid or contain $[4, 2, 3, 1]$?

$[1, 27, 2, 28, 3, 29, 4, 30, 5, 31, 6, 32, 7, 33, 8, 34, 9, 35, 10, 36, 11, 37, 12, 38, 1$

Pattern Avoidance on Permutations

Thm.(Guillemot-Marx 2014) For every permutation $v \in S_k$ there exists an algorithm to test if $w \in S_n$ contains v which runs in linear time, $O(n)$.

Pattern Avoidance on Permutations

Many important families of permutations have a pattern avoidance characterization and/or enumerative formulas.

1. Stack sortable permutations: 231 avoiding
2. Vexillary permutations: 2143 avoiding.
3. Smooth permutations: 3412 and 4231 avoiding.
4. Permutation patterns determine the irreducible components of singular loci of Schubert varieties.
5. Schubert varieties "defined by inclusions": 4231, 35142, 42513, 351624.
6. Deodhar permutations/ 321-hexagon avoiding: 321, 56781234, 56718234, 46781235, 46718235.
7. Boolean permutations: 321 and 3412 avoiding.
8. Factorial Schubert varieties: 4231 and 3412 avoiding.

Pattern Avoidance on Permutations

Def. A variety X is *factorial* at a point $\iff$ the local ring at that point is a unique factorization domain.

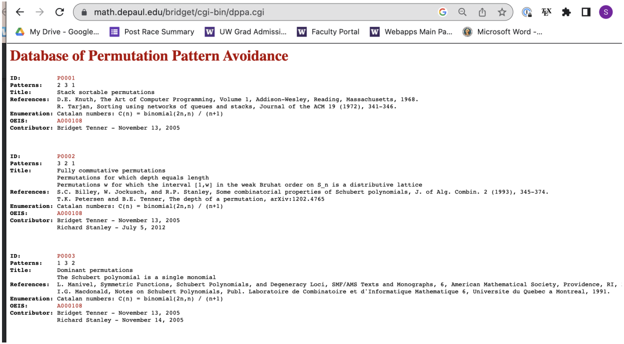
Thm. (Bousquet-Mélou+Butler, 2007, conj. by Woo-Yong) X_w is factorial at every point $\iff$ w avoids 4231 and $3\underline{41}2$. Here $3\underline{41}2$ means the 4 and 1 must be adjacent.

Thm.(Bousquet-Mélou+Butler) There is an explicit formula for counting the number f_n of factorial Schubert varieties for $w \in S_n$:

$$F(t) = \frac{(1-t)(1-4t-2t^2)-(1-5t)\sqrt{(1-4t)}}{2(1-5t+2t^2-t^3)}$$
$$= x + 2x^2 + 6x^3 + 22x^4 + 89x^5 + 379x^6 + 1661x^7 + 7405x^8 + \ldots$$

Pattern Avoidance on Permutations

Bridget Tenner's Database of Permutation Patterns has 61 families!



math.depaul.edu/bridget/cgi-bin/dppa.cgi

My Drive - Google... Post Race Summary UW Grad Admissi... Faculty Portal Webapps Main Pa... Microsoft Word -...

Database of Permutation Pattern Avoidance

ID: P0001
Patterns: 2 3 1
Title: Stack sortable permutations
References: D.E. Knuth, The Art of Computer Programming, Volume 1, Addison-Wesley, Reading, Massachusetts, 1968.
R. Tarjan, Sorting using networks of queues and stacks, Journal of the ACM 19 (1972), 341-346.
Enumeration: Catalan numbers: C(n) = binomial(2n,n) / (n+1)
OEIS: A000108
Contributor: Bridget Tenner - November 13, 2005

ID: P0002
Patterns: 3 2 1
Title: Fully commutative permutations
Permutations for which depth equals length
Permutations w for which the interval [1,w] in the weak Bruhat order on S_n is a distributive lattice
References: S.C. Billey, W. Jockusch, and R.P. Stanley, Some combinatorial properties of Schubert polynomials, J. of Alg. Combin. 2 (1993), 345-374.
T.K. Petersen and B.E. Tenner, The depth of a permutation, arXiv:1202.4765
Enumeration: Catalan numbers: C(n) = binomial(2n,n) / (n+1)
OEIS: A000108
Contributor: Bridget Tenner - November 13, 2005
Richard Stanley - July 5, 2012

ID: P0003
Patterns: 1 3 2
Title: Dominant permutations
The Schubert polynomial is a single monomial
References: L. Manivel, Symmetric Functions, Schubert Polynomials, and Degeneracy Loci, SMF/AMS Texts and Monographs, 6, American Mathematical Society, Providence, RI,
I.G. Macdonald, Notes on Schubert Polynomials, Publ. Laboratoire de Combinatoire et d'Informatique Mathematique 6, Universite du Quebec a Montreal, 1991.
Enumeration: Catalan numbers: C(n) = binomial(2n,n) / (n+1)
OEIS: A000108
Contributor: Bridget Tenner - November 13, 2005
Richard Stanley - November 14, 2005

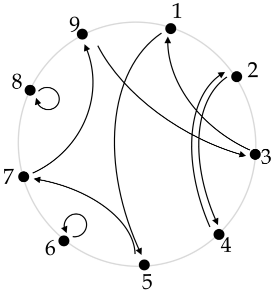
The International Conference on Permutation Patterns 2023 will take place at the University of Burgundy located in Dijon, France!

Decorated Permutations

Defn. A *decorated permutation* $w^{\circlearrowright}$ on n elements is a permutation $w \in S_n$ together with an orientation clockwise or counterclockwise assigned to each fixed point of w, denoted $\overrightarrow{i}$ or $\overleftarrow{i}$ respectively.

$$w^{\circlearrowright} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 1 & 2 & 7 & \overleftarrow{6} & 9 & \overrightarrow{8} & 3 \end{bmatrix} = 54127\overleftarrow{6}9\overrightarrow{8}3$$

Chord Diagram $D(w^{\circlearrowright})$



Decorated Permutations

Sylvie Corteel studied the q-analogs of Eulerian numbers related to the number of alignments and weak exceedances of permutations, which come from the enumeration of totally positive Grassmann cells following work of Williams and Postnikov.

Question. (Corteel, 2006) Can we define generalized patterns for decorated permutations?

Question.(Weaver, 2018) Are there interesting families of decorated permutations avoiding certain generalized patterns?

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Question.(Weaver, 2018) Are there interesting families of decorated permutations avoiding certain generalized patterns?

Answer. Yes! The ones indexing smooth positroid varieties.

Grassmannian Varieties

Notation.

- Fix $0 \leq k \leq n$.
- Mat_{kn} = the set of full rank $k \times n$ matrices over $\mathbb{C}$.
- $Gr(k,n)$ = *Grassmannian variety* of k-dim subspaces in $\mathbb{C}^n$
- $Gr(k,n) \approx \mathrm{GL}_k \backslash \mathrm{Mat}_{kn}$.
- $\Delta_J(M)$ = determinant of the $[k] \times J$ minor of M for $J \in \binom{[n]}{k}$.

The Grassmannian varieties are smooth manifolds via the *Plücker coordinate* embedding $M \mapsto (\Delta_J(M) : J \in \binom{[n]}{k})$ of $Gr(k,n)$ into projective space, including the case $k = n = 0$.

Plücker coordinates

$$\begin{bmatrix} 0 & 3 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \mapsto [0:0:0:0:0:0:3:6:3:1:2:1:0:2:4] \in \mathbb{P}^{15}$$

Nonvanishing coordinates at exactly the coordinates

$$\{\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,6\},\{5,6\}\} \subseteq \binom{[6]}{2}.$$

Def. The *matroid* of $A \in \mathrm{Mat}_{kn}$ is determined by the set of *bases*

$$\mathcal{M}_A = \{J \in \binom{[n]}{k} \mid \Delta_J(A) \neq 0\}.$$

or by the *non-bases*

$$\mathcal{Q}_A = \{J \in \binom{[n]}{k} \mid \Delta_J(A) = 0\}.$$

Positroids

Defn. (Postnikov) A *positroid* of rank k on ground set $[n]$ is a matroid of the form $\mathcal{M}_A$ for a matrix $A \in \mathrm{Mat}_{kn}$ such that every nonzero Plücker coordinate $\Delta_J(A)$ is positive.

Thm. (Postnikov 2006 + Oh 2011) There are bijections

1. positroids $\mathcal{M}$ of rank k on a ground set of size n,
2. decorated permutations w° on n elements with k anti-exceedances,
3. Grassmann necklaces $\mathcal{N} = (I_1, \ldots, I_n) \in \binom{[n]}{k}^n$, and
4. Grassmann intervals $[u, v]$ in $Gi(k, n)$.

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Important in the theory of the totally positive part of the Grassmannian variety, cluster algebras, and soliton solutions to the KP equations and have connections to statistical physics, integrable systems, and scattering amplitudes.

See Williams.ICM.2022, along with Lusztig.1994, Rietsch.1998, Fomin-Zelevinsky.2002, Kodama-Williams.2015, AHBCGPT.2016

Positroid Varieties

Defn.(Rietsch, Postnikov, Knutson-Lam-Speyer)
Given a decorated permutation $w^{\circ} \in S^{\circ}_{n,k}$ along with its associated Grassmann interval $[u, v]$ and positroid $\mathcal{M} = \mathcal{M}(w^{\circ}) \subseteq \binom{[n]}{k}$, the *positroid variety* $\Pi_{w^{\circ}} = \Pi_{[u,v]} = \Pi_{\mathcal{M}}$ is the subvariety of $Gr(k, n)$ with vanishing ideal generated by the Plücker coordinates indexed by the nonbases of $\mathcal{M}$, $\{\Delta_J : J \notin \mathcal{M}\}$.

Thm.(Knutson-Lam-Speyer) The positroid variety $\Pi_{[u,v]}$ is the projection of the Richardson variety $X_u^v \subseteq \mathcal{F}\ell(n)$ to $Gr(k, n)$.

Positroid Varieties

The vanishing coordinates of

$$A = \begin{bmatrix} 0 & 3 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \mapsto [0:0:0:0:0:0:3:6:3:1:2:1:0:2:4]$$

are exactly the coordinates indexed by the nonbases of $\mathcal{M}_A$

$$\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{4,5\}\} \subseteq \binom{[6]}{2}.$$

Therefore, the points in the positroid variety $\Pi_{\mathcal{M}_A}$ are represented by the full rank complex matrices of the form

$$\begin{bmatrix} 0 & a_{12} & ca_{12} & a_{14} & da_{14} & a_{16} \\ 0 & a_{22} & ca_{22} & a_{24} & da_{24} & a_{26} \end{bmatrix}.$$

Smooth versus singular varieties

Def. A variety X is *singular* at a point $x \in X$, if the dimension of the tangent space to X at x is strictly larger than the dimension of X. If no such point exists, X is *smooth*.

Def. If $X = Var(f_1, \ldots, f_s)$, the *Jacobian matrix*, *Jac*, is the matrix of partial derivatives of the f_i at each of the possible variables.

Def. The $\operatorname{rank}(Jac|_x)$ is the codimension of the tangent space to X at the point x.

Characterizing Smooth Positroid Varieties

Thm. (Billey-Weaver 2023) Let $\mathcal{M}$ be a rank k positroid on $[n]$ with associated decorated permutation $w^{\circlearrowleft}$, and Grassmann interval $[u, v]$. Then, the following are equivalent.

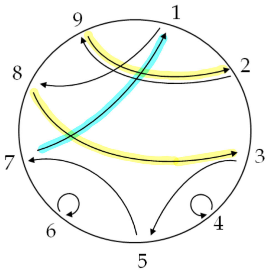
1. The positroid variety $\Pi_{w^{\circlearrowleft}} = \Pi_{[u,v]} = \Pi_{\mathcal{M}}$ is smooth.
2. The decorated permutation $w^{\circlearrowleft}$ has no crossed alignments.
3. The chord diagram $D(w^{\circlearrowleft})$ is a disjoint union of spirographs.
4. The positroid $\mathcal{M}$ is a direct sum of uniform matroids.

Some Patterns in Decorated Permutations

Patterns using chord diagrams.

- *alignments*: two directed edges in same direction
- *misalignments*: two directed edges in opposite direction
- *crossings*: two edges that must have a common point
- *crossed alignment*: an alignment plus a third edge crossing both sides

Consider the decorated permutation $w^{\circ} = 895\overleftarrow{4}7\overrightarrow{6}132$

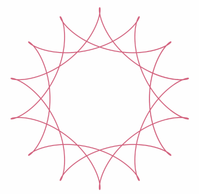
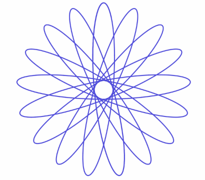
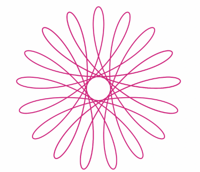


The highlighted crossed alignment pattern implies $\Pi_{w^{\circ}}$ is singular.

Patterns in Decorated Permutations

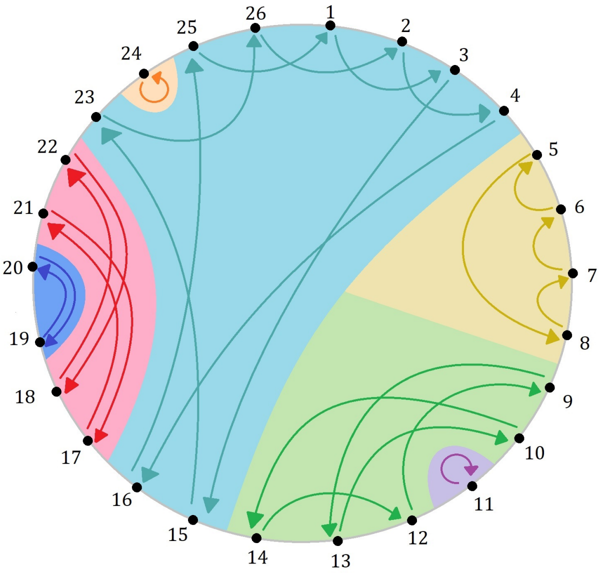
Patterns.

- *spirograph*: chord diagram of a connected decorated perm with $w(i) \mapsto i+m$ for all $i \in [n]$.

Thx to "Spirographs" made by the Spirograph Maker app for the iphone.

Chord diagram of a smooth positroid variety



Decorated Permutations to Grassmann intervals

Defn. The *anti-exceedance set* of w° is

$$I_1(w^{\circ}) = \{ i \in [n] \mid i < w^{-1}(i) \text{ or if } w^{\circ}(i) = \overrightarrow{i} \}$$

If $i \in [n]$ is not an anti-exceedance, it is an *exceedance*. Let $S^{\circ}_{n,k}$ be the set of decorated permutations on n elements with k anti-exceedances.

$$w^{\circ} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 1 & 2 & 7 & \overrightarrow{6} & 9 & \overleftarrow{8} & 3 \end{bmatrix} \Longrightarrow I_1(w^{\circ}) = \{1, 2, 6, 3\}.$$

Shuffling the anti-exceedances to the front bijectively determines the *Grassmann interval* $[u, v]$ associated to $w^{\circ} \in S^{\circ}_{n,k}$, where $u \leq v$ in Bruhat order on S_n and v is *k-Grassmannian*, so $v_1 < v_2 < \cdots < v_k$ and $v_{k+1} < \cdots < v_n$.

$$\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} 3 & 4 & 6 & 9 & 1 & 2 & 5 & 7 & 8 \\ 1 & 2 & 6 & 3 & 5 & 4 & 7 & 9 & 8 \end{bmatrix}.$$

Decorated Permutations to Positroids

The *Grassmann necklace* for $w^{\circlearrowleft} = 54127\,\overrightarrow{6}\,9\,\overleftarrow{8}\,3$ is

$$(I_1, \dots, I_9) = (\{1236\}, \{2356\}, \{3456\}, \{1456\}, \{1256\}, \{1267\}, \{1267\}, \{1269\}, \{1269\}).$$

The *positroid* for $w^{\circlearrowleft}$ is

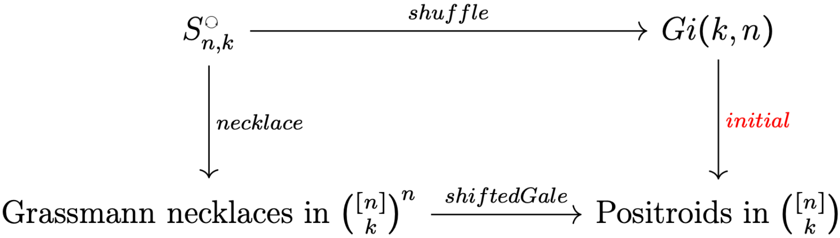
1236	1246	1256	1267	1269	1346	1356
1367	1369	1456	1467	1469	2356	2367
2369	2456	2467	2469	3456	3467	3469

Here, for each $r \in [n]$, let $r <_r <_r r+1 < \cdots n <_r 1 <_r < \cdots <_r r-1$,

$$I_r(w^{\circlearrowleft}) = \{i \in [n] \mid i <_r w^{-1}(i) \text{ or if } w^{\circlearrowleft}(i) = \overrightarrow{i}\},$$

$$\mathcal{M}(w^{\circlearrowleft}) := \left\{ I \in \binom{[n]}{k} : I_r(w^{\circlearrowleft}) \leq_r I \text{ for all } r \in [n] \right\}$$

Commutative Diagram of Bijections



Grassmann intervals to Positroids via initial sets.

$$\mathcal{M} = \{y[k] \;:\; y \in [u, v]\}.$$

Characterizing Smooth Positroid Varieties

Thm. (Billey-Weaver 2023) Let $\mathcal{M}$ be a rank k positroid on $[n]$ with associated decorated permutation $w^{\circlearrowleft}$, and Grassmann interval $[u, v]$. Then, the following are equivalent.

1. The positroid variety $\Pi_{w^{\circlearrowleft}} = \Pi_{[u,v]} = \Pi_{\mathcal{M}}$ is smooth.
2. The decorated permutation $w^{\circlearrowleft}$ has no crossed alignments.
3. The chord diagram $D(w^{\circlearrowleft})$ is a disjoint union of spirographs.
4. The positroid $\mathcal{M}$ is a direct sum of uniform matroids.

Outline of Proof

Thm. [Postnikov, KLS]

$$\mathrm{codim}(\Pi_{w^{\circlearrowleft}}) = \#\mathit{Alignments}(w^{\circlearrowleft}) = k(n-k) - [\ell(v) - \ell(u)].$$

Corollary. A positroid variety $\Pi_{w^{\circlearrowleft}}$ is a singular at x if and only if

$$\mathrm{rank}(Jac|_x) < \mathrm{codim}\,\Pi_{w^{\circlearrowleft}} = \#\mathit{Alignments}(w^{\circlearrowleft}).$$

Outline of Proof

By Construction.: For every $A \in \Pi_{\mathcal{M}}$, the first nonzero Plücker coordinate in lex order of A is in $\mathcal{M}$. For every $J \in \mathcal{M}$, the matrix which is the identity in columns J, denoted $A_J \in \Pi_{\mathcal{M}}$.

Thm. Assume $A \in \Pi_{\mathcal{M}}$ and $J \in \mathcal{M}$ indexes its first nonzero Plücker coordinate in lex order. Then the codimension of the tangent space to $\Pi_{\mathcal{M}}$ at A is bounded below by

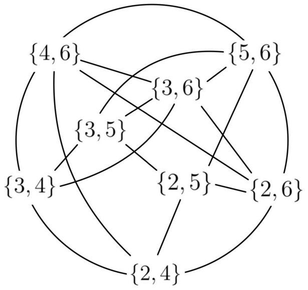
$$\text{rank}(Jac|_{A_J}) = \#\left\{ I \in \binom{[n]}{k} \smallsetminus \mathcal{M} \ : \ |I \cap J| = k-1 \right\}.$$

Hard Step.: Given a starboard tacking crossed alignment based at 1, the set on RHS maps injectively into the set of alignments of $w^{\circlearrowleft}$ and for $J = I_1(w^{\circlearrowleft}) \in \mathcal{M}$ there is at least one alignment that is not in the image.

Johnson Graphs

The *Johnson graph* $J(k, n)$ has vertices labeled by k-subsets of $[n]$ and two k-subsets I, J are connected by an edge precisely if $|I \cap J| = k - 1$.

For a positroid $\mathcal{M} \subseteq \binom{[n]}{k}$, let $J(\mathcal{M})$ denote the induced subgraph of the Johnson graph on the vertices in $\mathcal{M}$. We call $J(\mathcal{M})$ the *Johnson graph of* $\mathcal{M}$.



Regular Johnson Graphs and Smooth Positroid Varieties

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2. The decorated permutation w° has no crossed alignments.
3. The chord diagram $D(w^{\circ})$ is a disjoint union of spirographs.
4. The positroid $\mathcal{M}$ is a direct sum of uniform matroids.
5. The graph $J(\mathcal{M})$ is regular, and each vertex has degree $\ell(v) - \ell(u)$.

Enumerative Results

Similar to Ardila-Rincón-Williams enumeration of positroids via connected positroids, we use a theorem of Speicher to enumerate all smooth positroids via connected smooth positroids (aka spirograph permutations).

Thm. The number of smooth positroids on ground set $[n]$ is the coefficient

$$s(n) = [x^n]\frac{1}{n+1}\left(1 + 2x + \sum_{i=2}^{n}(i-1)x^i\right)^{n+1}. \tag{1}$$

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$$\begin{aligned} s(51)/s(50) &\approx 5.4489775, \\ s(101)/s(100) &\approx 5.528236, \\ s(151)/s(150) &\approx 5.555362, \\ s(201)/s(200) &\approx 5.569062, \\ s(251)/s(250) &\approx 5.5773263. \end{aligned}$$

Based on this data, we conjectured $s(n) \approx O(c^n)$ for $c \leq 6$.

Enumerative Results

Let $f(x) = x^{n+1}$ and let $g(x) = 1 + 2x + \sum_{i=2}^{n}(i-1)x^i$, then

$$s(n) = \frac{1}{n+1}[x^n]f(g(x)).$$

So take $\frac{d^n}{dx^n}f(g(x))$, set $x = 0$ and divide by $n+1$.

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So take $\frac{d^n}{dx^n}f(g(x))$, set $x = 0$ and divide by $n+1$.

Faá di Bruno Formula. (1855) Generalizing the chain rule from Calculus,

$$\frac{d^n}{dx^n}f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k}(g'(x), g''(x), \ldots, g^{(n+1-k)}(x)) \tag{2}$$

where *partial Bell polynomial* $B_{n,k}(x_1, \ldots, x_{n-k+1})$ is defined as

$$B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{B_1 \sqcup \cdots \sqcup B_k} \prod_{i=1}^{k} x_{|B_i|}, \tag{3}$$

where the sum is taken over all set partitions of $[n]$ into k blocks.

Enumerative Results

Notation. For $1 \leq k \leq n$, let $b_{0,0} = 1$, and $b_{0,k} = b_{n,0} = 0$ if $n > 0$ or $k > 0$,

- $B_{n,k}(x_1, \ldots, x_{n-k})$ Bell polynomial
- $b_{n,k} = B_{n,k}(2 \cdot 1!, 1 \cdot 2!, 2 \cdot 3!, \ldots, (n-k) \cdot (n-k+1)!)$

Cor. The number of smooth positroids on ground set $[n]$ is

$$s(n) = \sum_{k=1}^{n} \frac{b_{n,k}}{(n-k+1)!}. \tag{4}$$

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Cor. The number of smooth positroids on ground set $[n]$ with k connected components is $s_2(n,k) = \frac{b_{n,k}}{(n-k+1)!}$.

Enumerative Results

Thm.(Krattenthaler) For all non-negative integers n, the number of smooth positroids on the ground set $[n]$ is given by

$$s(n) = \frac{1}{n+1} \sum_{r=0}^{\lfloor (n+1)/2 \rfloor} (-1)^r 2^r \binom{n+1}{r} \binom{3n-3r+1}{n-2r}. \tag{5}$$

Thm.(Krattenthaler) The sequence $(s(n))_{n \geq 0}$ satisfies the recurrence relation

$$\begin{aligned} &2(n-2)(n+1)(2n+1)(26n-33)s(n) \\ &\quad - (1118n^4 - 3343n^3 + 2092n^2 + 367n - 66)s(n-1) \\ &\quad + 2(n-1)(2002n^3 - 6181n^2 + 4435n + 18)s(n-2) \\ &\quad - 4(n-2)(n-1)(1586n^2 - 3547n + 555)s(n-3) \\ &\quad + 152(n-3)(n-2)(n-1)(26n-7)s(n-4) = 0, \end{aligned} \tag{6}$$

with initial conditions $s(0) = 1$, $s(1) = 2$, $s(2) = 5$, and $s(3) = 16$.

Enumerative Results

Thm.(Krattenthaler) The asymptotic growth of the number of smooth positroids on the ground set $[n]$ is given by

$$s(n) \sim \frac{\rho^{n+1}}{n^{3/2}\sqrt{2\pi\xi}} = \frac{(5.61071\ldots)^{n+1}}{(8.74042\ldots)\cdot n^{3/2}}, \quad \text{as } n \to \infty, \qquad (7)$$

where $\rho = 5.61071\ldots$ is the real root of

$$4z^3 - 35z^2 + 84z - 76 = 0,$$

and $\xi = 12.15864\ldots$ is the real root of

$$38z^3 - 425z^2 - 416z - 416 = 0.$$

Enumerative Results

Thm.(Krattenthaler) For all integers n and k with $n \geq k > 0$, the number of smooth decorated permutations in $S_n^{\circlearrowleft}$ with exactly k connected components is given by

$$s(n,k) = \frac{1}{n+1}\binom{n+1}{k}\sum_{l=0}^{k} 2^{k-l}\binom{k}{l}\binom{n-k+l-1}{n-k-l}. \tag{8}$$

Future Work

Open problems.

1. Study the geometry and equivariant cohomology of a positroid variety via the induced directed Johnson graph on $\mathcal{M}$.

2. Identify the singular locus of a positroid variety.

3. Study the enumeration and coset structure of the group operations of flip, inverse, rotation for permutations, or for derangements, or SIF perms.

4. Study the combinatorics of the directed Johnson graphs of positroids.

Thank you!