Combinatorial Properties of Permutation Arrays (work in progress)

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Outline

1. Flags and Permutation Arrays
2. Flags in General Position
3. A Matroid Structure On The Generic Permutation Array
4. A Curious Connection To Tilings and Triangulations of $\boldsymbol{\Delta}^{d-1} \times \boldsymbol{\Delta}^{n-1}$
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Flags and Permutation Arrays

Defn. A complete flag in $\mathbb{C}^n$ is a nested sequence of vector spaces

$$F_\bullet = 0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n$$

Analogy.

Hyperplane Arrangements	$\sim$	Flag arrangements
$\mathcal{L}_A =$ intersection lattice	$\sim$	intersection table
$H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}$		$F^{(1)}_{x_1} \cap \cdots \cap F^{(d)}_{x_d}$

Flags and Permutation Arrays

Observation. For any 2 flags the intersection table is characterized by a permutation.

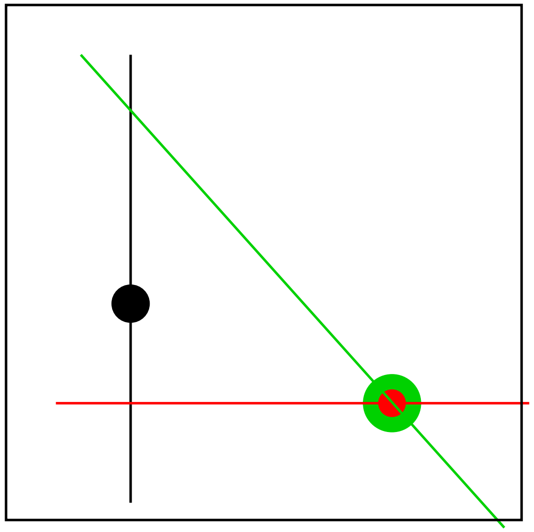
Example. $E_\bullet = \langle e_1, e_2, e_3 \rangle \qquad F_\bullet = \langle e_1 + e_2, e_1 - e_2, e_3 \rangle$

$$\begin{array}{c c} & \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{array}{c} F_1 \\ F_2 \\ F_3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \end{array} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in S_3$$

$$\begin{aligned} \dim E_i \cap F_j &= \#1's \text{ weakly north and west of position (i,j)} \\ &= \operatorname{rk} w[i,j]. \end{aligned}$$

Flags and Permutation Arrays

Observation. Given 3 flags $R_\bullet, B_\bullet, G_\bullet$ we can determine a 3-dimensional rank table for $\{\dim(R_i \cap B_j \cap G_j)\}$



G_1

	R_1	R_2	R_3
B_1			
B_2			
B_3	1	1	1

G_2

	R_1	R_2	R_3
B_1			
B_2			1
B_3	1	1	2

G_3

	R_1	R_2	R_3
B_1			1
B_2		1	2
B_3	1	2	3

$$\begin{aligned}
\dim B_i \cap R_j \cap G_k = \; & \#\text{ occupied rows weakly north, west, down of position (i,j,k)} \\
& \#\text{ occupied cols weakly north, west, down of position (i,j,k)} \\
& \#\text{ occupied boards weakly north, west, down of position (i,j,k)} \\
= \; & \mathbf{rk}\, P[(i,j,k)].
\end{aligned}$$

Flags and Permutation Arrays

Defn. (Eriksson-Linusson) For any subset $P \subset [n]^d$,

- $\mathrm{rk}_j P = \#\{k \mid \exists x \in P \text{ s.t. } x_j = k\}$.
- P is *rankable* of rank r if $\mathrm{rk}_j(P) = r$ for all $1 \leq j \leq d$.
- P is *totally rankable* if $P[x] = \{y \in P \mid y_i \leq x_i \forall i\}$ is rankable $\forall x$.
- P is a *permutation array* if it is totally rankable and has no redundant dots.

Theorem. (E-L) For every arrangement of d flags $E^1_\bullet, E^2_\bullet, \ldots, E^d_\bullet$, there exists a unique permutation array $P \subset [n]^d$ such that

$$\dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d}) = \mathrm{rk} P[x].$$

Hyperplane Arrangements vs Flag Arrangements

Analogy.

Hyperplane Arrangements	$\sim$	Flag arrangements
$\mathcal{L}_A =$ intersection lattice	$\sim$	intersection table
$H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}$		$F^{(1)}_{x_1} \cap \cdots \cap F^{(d)}_{x_d}$
matroids	$\sim$	permutation arrays

Open Problem. Is the realizability of permutations arrays equivalent to the realizability of matroids? Conjecture: Yes!.

Flags in general position

Easy Fact. The permutation array corresponding to d generically chosen flags $E^1_\bullet, \ldots, E^d_\bullet$ in $\mathbb{C}^n$ is given by $G_{n,d}$ where

$$G_{n,d} = \{x \in [n]^d \mid \sum x_i = (d-1)n + 1\}.$$

Example. $n = 4, d = 3$

$$G_{4,3} = \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline & & 4 & 3 \\ \hline & 4 & 3 & 2 \\ \hline 4 & 3 & 2 & 1 \\ \hline \end{array}$$

Observation. The elements of $G_{n,d}$ are in bijection with the 1-dim intersections

$$E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d}$$

Flags in General Position

Question. How can we characterize the dependent collections of vectors in $G_{n,d}$ using the permutation array?

Example. $n = 4, d = 3$

$$G_{4,3} = \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline & & 4 & \textcircled{3} \\ \hline & \textcircled{4} & \mathbf{3} & \textcircled{2} \\ \hline \textcircled{4} & \mathbf{3} & \mathbf{2} & 1 \\ \hline \end{array}$$

- Every principle subarray $G_{4,3}[x]$ corresponds to a collection of vectors of rank $\mathbf{rk}\ G_{4,3}[x]$.
- Circled numbers correspond with a set of vectors which should be independent. No k-subset is contained in a rank $k-1$ subarray.

Matroid Structure on $G_{n,d}$

Theorem. $G_{n,d}$ is the ground set of a matroid with independent sets given by the collection of subsets $I \subset G_{n,d}$ such that every parallel translate of $G_{k,d}$ contains at most k points of I for every $k \leq n$.

Example. $G_{3,3} = \begin{array}{|c|c|c|}\hline \cdot & \cdot & 3 \\ \hline \cdot & 3 & 2 \\ \hline 3 & 2 & 1 \\ \hline \end{array}$ has 5 symmetry classes of bases:

$$\begin{array}{|c|c|c|}\hline \cdot & \cdot & 3 \\ \hline \cdot & 3 & \cdot \\ \hline 3 & \cdot & \cdot \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline \cdot & \cdot & 3 \\ \hline \cdot & 3 & \cdot \\ \hline \cdot & 2 & \cdot \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline \cdot & \cdot & 3 \\ \hline \cdot & 3 & \cdot \\ \hline \cdot & \cdot & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline \cdot & \cdot & 3 \\ \hline \cdot & \cdot & \cdot \\ \hline 3 & \cdot & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline \cdot & \cdot & \cdot \\ \hline \cdot & 3 & 2 \\ \hline \cdot & 2 & \cdot \\ \hline \end{array}$$

e.g. $\begin{array}{|c|c|c|}\hline \cdot & \cdot & 3 \\ \hline \cdot & 3 & \cdot \\ \hline 3 & \cdot & \cdot \\ \hline \end{array}$ $\equiv$ orbit of $\{(1,3,3),(2,2,3),(3,1,3)\}$ under S_3.

Curious Connection To Lozenge Tilings

$G_{3,3}$ has 5 symmetry classes of bases:

$\cdot$	$\cdot$	**3**
$\cdot$	**3**	$\cdot$
3	$\cdot$	$\cdot$

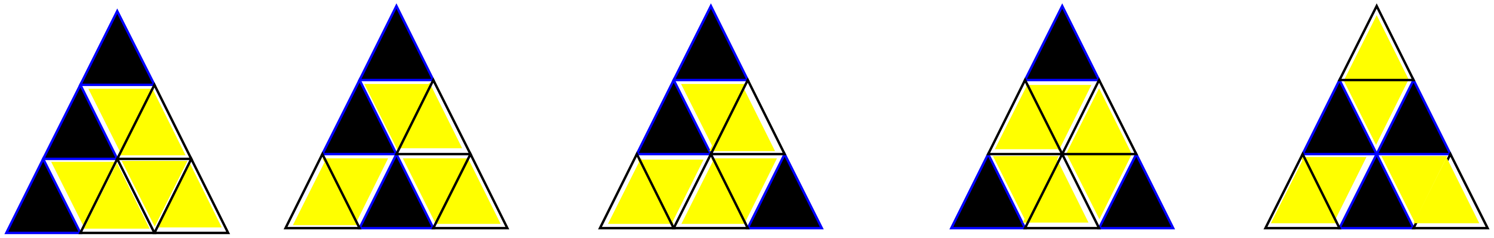
$\cdot$	$\cdot$	**3**
$\cdot$	**3**	$\cdot$
$\cdot$	**2**	$\cdot$

$\cdot$	$\cdot$	**3**
$\cdot$	**3**	$\cdot$
$\cdot$	$\cdot$	**1**

$\cdot$	$\cdot$	**3**
$\cdot$	$\cdot$	$\cdot$
3	$\cdot$	**1**

$\cdot$	$\cdot$	$\cdot$
$\cdot$	**3**	**2**
$\cdot$	**2**	$\cdot$

Lozenge Tiling from "Caley Trick and Products of Simplices" by Francisco Santos:



Curious Connection To Lozenge Tilings

- $T(n)$ = equilateral triangle with side length n.
- Tile $T(n)$ using unit triangles and unit rhombi with angles 60° and 120°

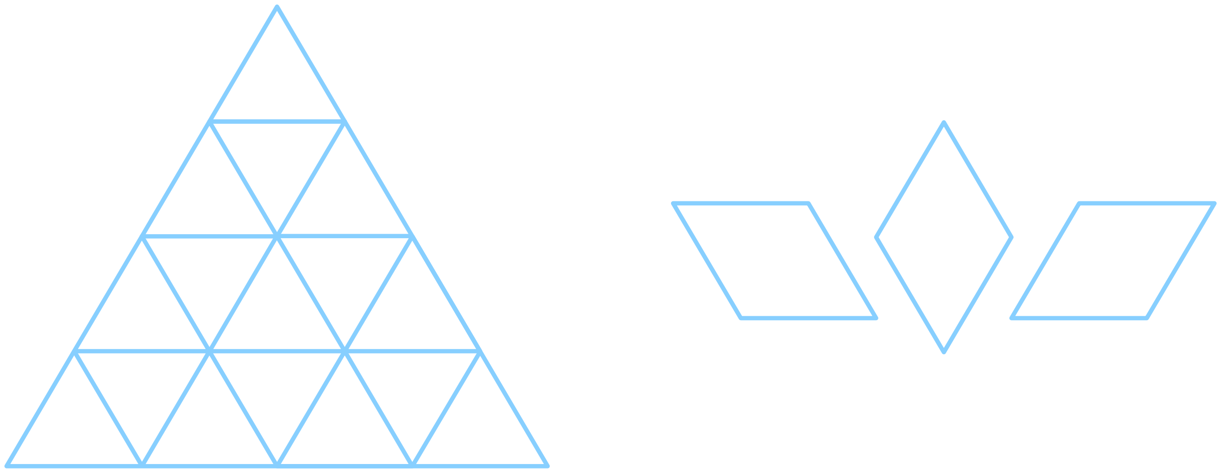


Figure 1: $T(4)$ and unit rhombi.

Observation. There must be exactly n unit triangles if a tiling exists.

$$\#\triangle - \#\triangledown = \binom{n+1}{2} - \binom{n}{2} = n$$

Curious Connection To Lozenge Tilings

Question. Given n holes in $T(n)$, is there a simple criterion to determine whether there exists a rhombus tiling of the holey triangle remains?

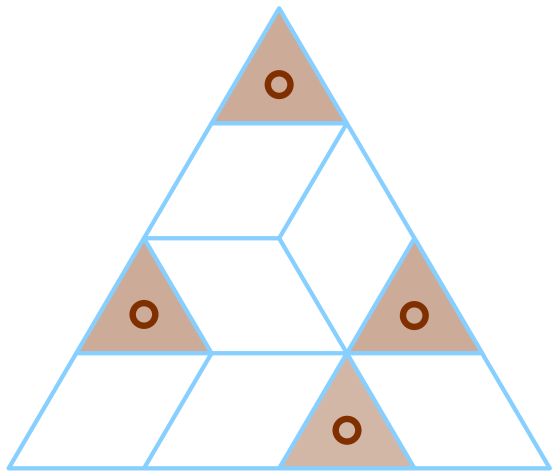


Figure 2: A tiling of a holey $T(4)$.

Curious Connection To Lozenge Tilings

Theorem. The triangle $T(n)$ with n fixed holes removed can be tiled with rhombi if and only if every $T(k)$ in $T(n)$ contains at most k of the holes, for all $k \leq n$.

Proof.

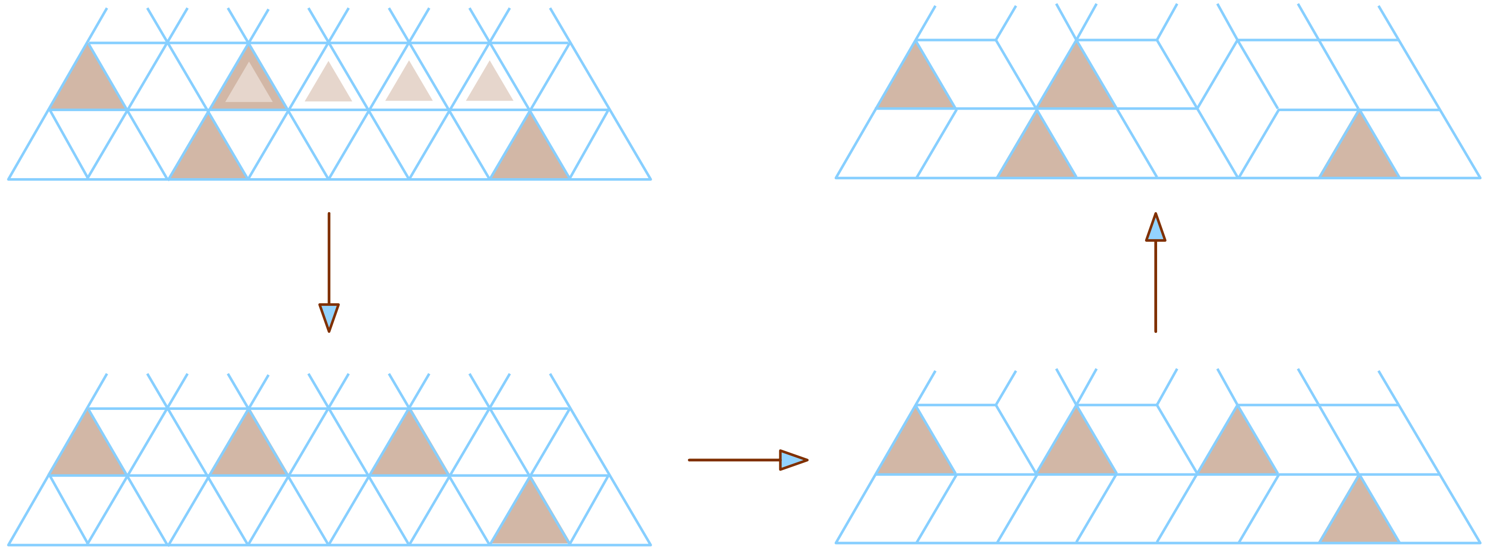


Figure 3: Sliding the hole across and up to next row.

Curious Connection To Lozenge Tilings

Corollary. The possible locations of n holes for which a rhombus tiling of the holey triangle $T(n)$ exists correspond to the bases of the matroid $G_{n,3}$.

Theorem. (F. Santos) The number of triangulations of $\Delta^2 \times \Delta^{n-1}$ equals $n!$ times the number of lozenge tilings of $T(n)$.

Open Question. Can we interpret the lozenge tilings of $T(n)$ as some geometric structure on 3 flags?

Triangulations of $\Delta^{d-1} \times \Delta^{n-1}$

Theorem. (Develin - Sturmfels "Tropical Convexity")
The number of combinatorially distinct generic tropical complexes generated by d points in $\mathbb{TP}^{n-1}$ equals the number of distinct regular triangulations of $\Delta^{d-1} \times \Delta^{n-1}$

Caley Trick. (due to Sturmfels, Huber) Bijection between polyhedral subdivisions of $\Delta^{d-1} \times \Delta^{n-1}$ and mixed subdivisions of the dilation $n\Delta^{d-1}$. For $d = 3$, mixed subdivisions of $n\Delta^2$ are lozenge tilings (Santos).

Conjecture. The possible locations of the n simplices in a mixed subdivision of $n\Delta^{d-1}$ correspond to the bases of the matroid $G_{n,d}$.

Summary of Open Questions

- How many permutations arrays are there in $[n]^d$?
- Is the realizability of permutations arrays equivalent to the realizability of matroids? Conjecture: Yes!.
- Can we find a matroid or flag interpretation for the lozenge tilings of the dilated triangle $T(n)$ coming from the Develin-Sturmfels/Santos triangulations of $\Delta^2 \times \Delta^{n-1}$?
- Is it true that the possible locations of the n simplices in a mixed subdivision of $n\Delta^{d-1}$ correspond to the bases of the matroid $G_{n,d}$.